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Lie symmetries of finite strain elastic–perfectly plastic models and exactly consistent schemes for numerical integrations

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Abstract

For finite strain elastic–perfectly plastic models, the use of non-spinning objective stress rates leads to cubic degree non-linear differential equations as well as the coupling of deviatoric and volumetric constitutive equations. The governing equations reduce to quadratic non-linearity and the coupling disappears when spinning objective stress rates are used. In order to effectively deal with these non-linearities we first convert the non-linear constitutive equations into a Lie type system, $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$, $\mathbf{A} \in so(5,1)$, for $\mathbf{X}$ in the Minkowski spacetime $\mathbb{M}^{5+1}$, which has an indefinite metric $\mathbf{g}$ of signature (5,1). However, for spinning objective stress rates $\mathbf{A}$ merely depends on t through deformation history and the above equation reduces to a time-varying linear system. The new representation admits a Lie symmetry of the proper orthochronous Lorentz group $SO_o(5,1)$ in the plastic phase. In terms of $\mathbf{X}$, the yield condition is transformed to a null cone condition $\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0$ in $\mathbb{M}^{5+1}$. Due to this nullity we thus assign a quaternionic Hermitian matrix to represent $\mathbf{X}$, and convert the six-dimensional system to a more economic lower dimensional quaternionic two-component spinor system, $\dot{\boldsymbol{\alpha}} = \mathbf{Q}\boldsymbol{\alpha}$, $\mathbf{Q} \in sl(2,\mathbb{H})$. In the two-dimensional spinor space the Lie symmetry is found to be $SL(2,\mathbb{H})$, the mapping of which onto $SO_o(5,1)$ is derived. In addition to the mapping between $so(5,1)$ and $sl(2,\mathbb{H})$, an isomorphic mapping between $sl(2,\mathbb{H})$ and $su^*(4)$ is also established, the latter of which generates the group $SU^*(4)$. Finally, according to these Lie symmetries the numerical schemes preserving the group properties are developed, which satisfy the consistency condition [1] exactly for every time step without iterations at all.

Keywords: Elastoplasticity; Finite strain models; Objective stress rates; Lorentz group; Lie symmetries; Quaternionic two-component spinor; Consistent schemes

1. Introduction

A novel formulation for elastoplasticity has been recently developed by Hong and Liu (1999a,b, 2000), Liu and Hong (2001), Liu (2001a, 2003, in press), and Mukherjee and Liu (2003). These authors have

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[1] The consistency means that during the plastic deformation the stress point must remain on the subsequent yield surface.

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explored the internal symmetry groups of the constitutive models for perfect elastoplasticity with or without considering large rotation, for bilinear elastoplasticity, for visco-elastoplasticity, for isotropic work-hardening elastoplasticity, as well as for mixed-hardening elastoplasticity to ensure that the plastic consistency condition is exactly satisfied at each time step once the computational scheme can take these symmetries into account.

In order to grasp the concepts that this paper needs we employ a small-strain elastic–perfectly plastic model as a demonstrated example:

$$\dot{\mathbf{s}} + \frac{G\dot{\lambda}}{\tau_y^0}\mathbf{s} = 2G\dot{\mathbf{e}}, \tag{1}$$

where $\dot{\lambda}$ subjects to the switching criteria

$$\dot{\lambda} = \begin{cases} \frac{1}{\tau_y^0}\mathbf{s}\cdot\dot{\mathbf{e}} > 0 & \text{if } \|\mathbf{s}\| = \sqrt{2}\tau_y^0 \quad \text{and} \quad \mathbf{s}\cdot\dot{\mathbf{e}} > 0, \\ 0 & \text{if } \|\mathbf{s}\| < \sqrt{2}\tau_y^0 \quad \text{or} \quad \mathbf{s}\cdot\dot{\mathbf{e}} \leqslant 0. \end{cases} \tag{2}$$

As usual $\mathbf{s}$ and $\mathbf{e}$ are respectively stress and strain deviators, τ_y^0 is the shear yield strength, and G is the shear modulus. While $\|\mathbf{s}\| := \sqrt{\mathbf{s}\cdot\mathbf{s}}$ defines the Euclidean norm of $\mathbf{s}$, and a dot between two same order tensors denotes their Euclidean inner product, $\|\mathbf{s}\| = \sqrt{2}\tau_y^0$ signifies the yield condition, $\mathbf{s}\cdot\dot{\mathbf{e}} > 0$ the loading condition, and $\mathbf{s}\cdot\dot{\mathbf{e}} \leqslant 0$ the unloading condition. Obviously the governing equation for $\mathbf{s}$ is non-linear in the plastic phase.

Let us introduce [2]

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 \\ X^4 \\ X^5 \\ X^0 \end{bmatrix} := \frac{X^0}{\tau_y^0}\begin{bmatrix} \beta_1 s^{11} + \beta_2 s^{22} \\ \beta_3 s^{11} + \beta_4 s^{22} \\ s^{23} \\ s^{13} \\ s^{12} \\ \tau_y^0 \end{bmatrix}, \tag{3}$$

where

$$X^0 := \exp\left(\frac{G\lambda}{\tau_y^0}\right) \tag{4}$$

is an internal time, and

$$\beta_1 := \sin\left(\theta + \frac{\pi}{3}\right), \quad \beta_2 := \sin\theta, \quad \beta_3 := \cos\left(\theta + \frac{\pi}{3}\right), \quad \beta_4 := \cos\theta \tag{5}$$

with θ being any real number. Then, the model represented by Eqs. (1) and (2) can be transformed into a time-varying linear system (see, e.g., Hong and Liu, 1997, 2000)

$$\dot{\mathbf{X}} = \mathbf{A}(t)\mathbf{X}, \tag{6}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_5 & \mathbf{A}_0^s \\ \mathbf{A}_s^0 & 0 \end{bmatrix} \quad \text{if } \mathbf{X}^T\mathbf{g}\mathbf{X} = 0 \quad \text{and} \quad \frac{d}{dt}[(\mathbf{X}^s)^T\mathbf{X}^s] > 0, \tag{7}$$

[2] A special case of Eq. (3) with $\theta = 0$ can be viewed as the homogeneous coordinates (non-dimensionalized with respect to the tensile yield strength $\sqrt{3}\tau_y^0$) of the Il'yushin stress space $(\frac{3}{2}s^{11}, \frac{\sqrt{3}}{2}s^{11} + \sqrt{3}s^{22}, \sqrt{3}s^{23}, \sqrt{3}s^{13}, \sqrt{3}s^{12})$.

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_5 & \mathbf{A}_0^{\mathrm{s}} \\ \mathbf{0}_{1\times 5} & 0 \end{bmatrix} \quad \text{if } \mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} < 0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^{\mathrm{s}})^{\mathrm{T}}\mathbf{X}^{\mathrm{s}}] \leqslant 0, \tag{8}$$

in which

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_5 & \mathbf{0}_{5\times 1} \\ \mathbf{0}_{1\times 5} & -1 \end{bmatrix}, \tag{9}$$

$$(\mathbf{A}_{\mathrm{s}}^0)^{\mathrm{T}} = \mathbf{A}_0^{\mathrm{s}} := \frac{2G}{\tau_{\mathrm{y}}^0} \begin{bmatrix} \beta_1 \dot{e}_{11} + \beta_2 \dot{e}_{22} \\ \beta_3 \dot{e}_{11} + \beta_4 \dot{e}_{22} \\ \dot{e}_{23} \\ \dot{e}_{13} \\ \dot{e}_{12} \end{bmatrix}. \tag{10}$$

$\mathbf{0}_5$ and $\mathbf{I}_5$ denote, respectively, the five-order zero and identity matrices, and the superscript 'T' denotes the transpose.

For the formulation represented by Eqs. (3)–(10) there have several points deserved to point out:

(a) The variable $\mathbf{X}$ defined in Eq. (3) that includes the internal time X^0 as temporal component and the other five independent deviatoric stresses multiplied by X^0/τ_{y}^0 as spatial component, $\mathbf{X}^{\mathrm{s}}$, may be called *augmented stress*. The underlying space of $\mathbf{X}$ denoted by $\mathbb{M}^{5+1}$ is called the *Minkowski spacetime*, whose metric tensor $\mathbf{g}$ as given by Eq. (9) is of signature (5,1).
(b) A deviatoric stress point $\mathbf{s}$ on the yield surface $\|\mathbf{s}\| = \sqrt{2}\tau_{\mathrm{y}}^0$ in the Euclidean space $\mathbb{E}^5$ corresponds to an augmented stress point $\mathbf{X}$ on the right circular cone $\{\mathbf{X}|\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0\}$ emanating from $\mathbf{X} = \mathbf{0}$ in the Minkowski spacetime $\mathbb{M}^{5+1}$, while an $\mathbf{s}$ within the yield surface corresponds to an $\mathbf{X}$ in the interior $\{\mathbf{X}|\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} < 0\}$ of the cone. $\mathbf{X} \in \mathbb{M}^{5+1}$ is called a *null vector* if it satisfies the cone condition $\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0$.
(c) The loading condition $\mathbf{s} \cdot \dot{\mathbf{e}} > 0$ and the unloading condition $\mathbf{s} \cdot \dot{\mathbf{e}} \leqslant 0$ are changed to $\mathrm{d}[(\mathbf{X}^{\mathrm{s}})^{\mathrm{T}}\mathbf{X}^{\mathrm{s}}]/\mathrm{d}t > 0$ and $\mathrm{d}[(\mathbf{X}^{\mathrm{s}})^{\mathrm{T}}\mathbf{X}^{\mathrm{s}}]/\mathrm{d}t \leqslant 0$, respectively.
(d) For the linear system (6), whose $\mathbf{A}$ in the plastic phase as given in Eq. (7) satisfies

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}. \tag{11}$$

The set of all $(5+1)\times(5+1)$ matrices that satisfy the above relation is denoted by $so(5,1)$.
(e) A matrix function $\mathbf{G}(t)$ that satisfies

$$\dot{\mathbf{G}}(t) = \mathbf{A}(t)\mathbf{G}(t), \tag{12}$$

$$\mathbf{G}(0) = \mathbf{I}_6 \tag{13}$$

with $\mathbf{A}(t) \in so(5,1)$ is called a single-parameter *Lorentz group*. From Eqs. (11)–(13) it follows readily that

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}. \tag{14}$$

The $\mathbf{G}$ if further satisfying

$$\det \mathbf{G} = 1, \tag{15}$$

$$G_0^0 \geqslant 1 \tag{16}$$

is called the *proper orthochronous Lorentz group*, and is denoted by $SO_o(5,1)$. It is an internal symmetry group of the small-strain elastic–perfectly plastic model in the plastic phase. In the above det is the shorthand of determinant and G_0^0 is the 00th mixed component of $\mathbf{G}$.

It is known that an element $\mathbf{A}$ of the real Lie algebra $so(5,1)$ of the proper orthochronous Lorentz group $SO_o(5,1)$ has the general form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^s_s & \mathbf{A}^s_0 \\ \mathbf{A}^0_s & 0 \end{bmatrix}$$

with properties:

$$(\mathbf{A}^s_s)^T = -\mathbf{A}^s_s, \quad (\mathbf{A}^0_s)^T = \mathbf{A}^s_0.$$

But the form of $\mathbf{A}$ previously found by Hong and Liu (1997, 2000) as that shown in Eq. (7) for the small-strain deformation models of elastoplasticity has zero $\mathbf{A}^s_s$ and is, therefore, less general. As shown by Hong and Liu (1999a) and Liu and Hong (2001), we may generalize the model by including a non-vanishing, skew-symmetric tensor $\mathbf{A}^s_s$ in $\mathbf{A}$ if finite strain deformation and rotation are considered. The original version presented an extremely simple framework leading to a linear set of differential equations with obvious advantage for numerical integrations.

In this paper, we extend these works to large deformation constitutive equations with different corotational or non-corotational stress rates reported in the literature, and compare the behavior of these models under simple shear deformation. The numerical results about the simple shear behavior of the elastoplastic models with different objective stress rates are shown here for completeness. Some similar results have been examined and compared by Atluri (1984), Szabó and Balla (1989) and others. The large deformation version is not so simple, since it is highly non-linear but would be clear later that our extension still has many advantages for numerical algorithms.

The objective rates of Kirchhoff stress [3] are summarized by the form $\overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - 2[\mathbf{B}\boldsymbol{\tau}]$, where $\mathbf{B}$ is assigned and is function of t through deformation history. For example, the objective stress rates of Truesdell (1955), Oldroyd (1950), Cotter and Rivlin (1955), Jaumann (1911), Durban and Baruch (1977), Green and Naghdi (1965), Sowerby and Chu (1984), Szabó and Balla (1989), and Xiao et al. (1997a,b, 1999) are all of this form. Due to the appearance of spurious shear oscillatory behavior of hypoelastic models and elastoplastic models with some objective stress rates, considerable efforts have been made to resolve the problem of choosing an appropriate objective stress rate in rate-form constitutive equations (see, e.g., Bruhns et al., 2001; Xiao et al., 2001; Lin, 2003; and references therein). Liu and Hong (1999) have examined the above ten objective stress rates for the model of hypoelasticity, and obtained two criteria for oscillatory and non-oscillatory stress response under simple shear deformation. Then, Liu and Hong (2001) used the comparison theorem to derive a sufficient criterion for non-oscillatory stress response under simple shear of the elastoplastic models using corotational stress rates.

Here, we investigate the finite strain models of elastoplasticity by considering the decomposition of deformation rate with $\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p$ (see, e.g., Neale, 1981; Nemat-Nasser, 1982; Bruhns et al., 1999) and by employing the above ten objective stress rates on the constitutive equations. We attempt to unify the finite strain theory from the point view of the Minkowski spacetime and its admissible internal symmetry group. Furthermore, we also utilize the Lie group and its Lie algebra properties to develop high accuracy and high efficiency schemes, which can be used to calculate the stress response under different deformation conditions. In the plastic phase the augmented stress, which is a null vector in space $\mathbb{M}^{5+1}$, can be mathematically identified to a 2×2 quaternionic Hermitian matrix with zero determinant. Then we can develop a quaternion based spinor representation of the six-dimensional proper orthochronous Lorentz group, as well as a two-component spinor framework for the plasticity theory with finite strain deformation.

As we know the numerical schemes developed up to now for the integration of constitutive equations of elastoplasticity are executed directly in stress space, for example, the tangent stiffness-radial return method (see, e.g., Krieg and Krieg, 1977; Schreyer et al., 1979), the radial return method (see, e.g., Krieg and Krieg, 1977; Kobayashi and Ohno, 2002; Lubarda and Benson, 2002; Auricchio and Beirão da Veiga, 2003), the

[3] In the literature of finite strain plasticity we can find the so-called Kirchhoff stress tensor $\boldsymbol{\tau} = \boldsymbol{\sigma} \det \mathbf{F}$, where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, and $\mathbf{F}$ the deformation gradient tensor.

elastic predictor–radial corrector method (see, e.g., Schreyer et al., 1979), the generalized midpoint rule (see, e.g., Ortiz and Popov, 1985), the closest-point-projection algorithm (see, e.g., Simo and Taylor, 1985), and also the plastic predictor–elastic corrector method (see, e.g., Nemat-Nasser, 1991). In order to enforce the consistency condition at every time step the above-mentioned algorithms require some iterative calculations until stress point at the end of each time step converges to the yield surface (see, e.g., Simo and Hughes, 1998), which is known as a main source of numerical errors and of consumption of computational time. A numerical scheme which preserves symmetry and utilizes the invariance property will be more capable of capturing key features during elastoplastic deformation and has long-term stability and much more improved efficiency and accuracy. Therefore the issue of internal symmetries in the constitutive laws of plasticity is not only important in its own right, but will also find applications to computational plasticity.

In this paper we consider finite strain constitutive models of perfect elastoplasticity with different objective stress rates in Sections 2 and 3, and manage to put them in a more appropriate setting in Section 4, such that the internal spacetime structure and the internal symmetries of the models are brought out in Section 5. Then, taking advantage of the nullity of the introduced augmented stress in plastic phase we develop a mathematically equivalent quaternionic Hermitian matrix representation in Section 6, which is suitable to derive the spinor map from $SL(2,\mathbb{H})$ onto $SO_o(5,1)$ via an extension of Dirac's method. Some properties about quaternion algebra $\mathbb{H}$ are summarized in Appendix A. In order to precisely derive the relations of the corresponding time-varying Lie algebras we develop a direct approach to realize the spinor map in Appendix B, and then the Lie algebras isomorphism in Appendix C. It will be sure that the newly proposed direct approach is rather crucial to pinpoint the relation between $sl(2,\mathbb{H})$ and $so(5,1)$. In addition to the spinor representation we also develop a $SU^*(4)$ group representation in Section 7. Then, by using these internal symmetries inherent in the constitutive models we develop consistent schemes in Section 8. One direct benefit of these schemes is that the stress point is exactly updated on the yield surface without iterative calculations for every time step. This is what the conventional numerical schemes desired and failed to achieve. Finally, we draw conclusions in Section 9.

As we merely consider the perfectly elastoplastic finite strain effect on material, which can be regarded as a prototype plastic mechanism indicating what might be achieved by this type finite strain modeling rather than as a definitive statement about the real material behavior, the other hardening mechanisms and elastoplastic models are intensively discussed in current research circles with the goal of more closely representing reality. To achieve comprehensive simulation of material behavior, it is necessary to use more realistic elastoplastic models taking both finite strain and hardening effects into account. Xiao et al. (2001) and Bruhns et al. (2001) have extended their self-consistent finite strain models with logarithmic rate to the kinematic hardening plasticity. Numerical tests with simple shear and torsion show that their models can obtain satisfactory and reasonable explanation of experimental observation. Although we have investigated the internal symmetry of small strain Prager kinematic hardening plasticity in Hong and Liu (1999b), and small strain mixed-hardening plasticity in Liu (in press), it deserves in the future to extend the current study on finite strain kinematic hardening plasticity and also on other finite strain elastoplastic models.

2. Large deformation constitutive models

The constitutive law of elastoplasticity of solid materials proposed by Prandtl and Reuss can be re-postulated and enlarged to take account of large deformation as in the following postulations:

$$\mathbf{D} = \mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{p}}, \tag{17}$$

$$\overset{\circ}{\boldsymbol{\tau}} = 2G\mathbf{D}^{\mathrm{e}} + \lambda(\operatorname{tr}\mathbf{D})\mathbf{I}_3, \tag{18}$$

$$\mathbf{s}\dot{\lambda} = 2\tau_y^0 \mathbf{D}^p, \tag{19}$$

$$\|\mathbf{s}\| \leqslant \sqrt{2}\tau_y^0, \tag{20}$$

$$\dot{\lambda} \geqslant 0, \tag{21}$$

$$\|\mathbf{s}\|\dot{\lambda} = \sqrt{2}\tau_y^0\dot{\lambda}, \tag{22}$$

where the Lamé material parameters G and λ and the shear yield stress τ_y^0 are the only three property constants needed in the isotropic hypoelastic–perfectly plastic large deformation model. It is postulated that $0 < G < \infty$, $-2G/3 < \lambda < \infty$ and $\tau_y^0 > 0$, and that $\operatorname{tr}\mathbf{D}^p = 0$. We let $\boldsymbol{\tau}$ denote the Kirchhoff stress and $\mathbf{s}$ its deviatoric part, i.e., $\mathbf{s} := \boldsymbol{\tau} - \mathbf{I}_3(\operatorname{tr}\boldsymbol{\tau})/3$, where tr denotes the trace of the tensor. The bold-faced symbols $\mathbf{D}$, $\mathbf{D}^e$ and $\mathbf{D}^p$ stand for the deformation rate, elastic deformation rate, and plastic deformation rate, respectively, all being symmetric tensors, whereas λ is a scalar, called the equivalent shear plastic strain.

A superimposed dot denotes the time derivative, that is d/dt, and a surmounted circle "∘" on $\boldsymbol{\tau}$ represents an objective stress rate (see, e.g., Szabó and Balla, 1989; Liu and Hong, 1999)

$$\overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - 2[\mathbf{B}\boldsymbol{\tau}], \tag{23}$$

where $[\cdot]$ means to take the symmetric part of its tensorial argument. When $\mathbf{B}$ takes different definitions, $\overset{\circ}{\boldsymbol{\tau}}$ stands for different objective rates as listed in Table 1.

The notations used in Table 1 are summarized as follows: $\mathbf{L} := \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the velocity gradient tensor, where $\mathbf{F}$ is the two-point tensor of deformation gradient; $\mathbf{D}$ and $\mathbf{W}$ are the symmetric and skew-symmetric parts of $\mathbf{L}$, respectively. $\boldsymbol{\Omega} := \dot{\mathbf{R}}\mathbf{R}^T$ is the rate of rotation, where $\mathbf{R}$ is the rotation tensor in the polar decomposition $\mathbf{F} = \mathbf{V}\mathbf{R}$ of $\mathbf{F}$. $\boldsymbol{\Omega}_E := \dot{\mathbf{R}}_E\mathbf{R}_E^T$ is known as the Eulerian spin tensor, where $\mathbf{R}_E$ is the diagonal transformation of $\mathbf{V}$, that is

$$\mathbf{V} = \mathbf{R}_E\boldsymbol{\lambda}\mathbf{R}_E^T, \tag{24}$$

with $\boldsymbol{\lambda} = \operatorname{diag}[\lambda_1, \lambda_2, \lambda_3]$ the diagonal tensor containing the eigenvalues, $\lambda_1, \lambda_2, \lambda_3$, of $\mathbf{V}$; $\mathbf{L}_E$ is defined by

$$\mathbf{L}_E := \dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\boldsymbol{\Omega}_E\mathbf{V}^{-1}.$$

The $\mathbf{B}$ is grossly referred to as the general spin if $\mathbf{B}^T = -\mathbf{B}$. Accordingly, the objective rates can be grouped into two main classes: *spinning* and *non-spinning* (or *corotational* and *non-corotational*). An objective stress rate $\overset{\circ}{\boldsymbol{\tau}}$ defined in Eq. (23) is said to be spinning if $\mathbf{B}^T = -\mathbf{B}$; otherwise, it is non-spinning. Table 1 lists 10 objective stress rates and their corresponding $\mathbf{B}$'s.

Table 1
10 objective stress rates and their symmetric $[\mathbf{B}]$'s and skew-symmetric $\langle\mathbf{B}\rangle$'s

Objective stress rates $\overset{\circ}{\boldsymbol{\tau}}$	$\mathbf{B}$	$\langle\mathbf{B}\rangle$	$[\mathbf{B}]$
1. Truesdell(T)	$\mathbf{L} - \frac{1}{2}(\operatorname{tr}\mathbf{D})\mathbf{I}_3$	$\mathbf{W}$	$\mathbf{D} - \frac{1}{2}(\operatorname{tr}\mathbf{D})\mathbf{I}_3$
2. Oldroyd (O)	$\mathbf{L}$	$\mathbf{W}$	$\mathbf{D}$
3. Cotter–Rivlin (CR)	$-\mathbf{L}^T$	$\mathbf{W}$	$-\mathbf{D}$
4. Jaumann (J)	$\mathbf{W}$	$\mathbf{W}$	$\mathbf{0}$
5. Durban–Baruch (DB)	$\frac{1}{2}\mathbf{D} + \mathbf{W} - \frac{1}{2}(\operatorname{tr}\mathbf{D})\mathbf{I}_3$	$\mathbf{W}$	$\frac{1}{2}\mathbf{D} - \frac{1}{2}(\operatorname{tr}\mathbf{D})\mathbf{I}_3$
6. Green–Naghdi (GN)	$\boldsymbol{\Omega}$	$\boldsymbol{\Omega}$	$\mathbf{0}$
7. Sowerby–Chu (SC)	$\boldsymbol{\Omega}_E$	$\boldsymbol{\Omega}_E$	$\mathbf{0}$
8. Szabó–Balla-1 (SB1)	$\mathbf{L}_E$	$\frac{1}{2}(\mathbf{L}_E - \mathbf{L}_E^T)$	$\frac{1}{2}(\mathbf{L}_E + \mathbf{L}_E^T)$
9. Szabó–Balla-2 (SB2)	$-\mathbf{L}_E^T$	$\frac{1}{2}(\mathbf{L}_E - \mathbf{L}_E^T)$	$\frac{-1}{2}(\mathbf{L}_E + \mathbf{L}_E^T)$
10. Xiao–Bruhns–Meyers (XBM)	$\boldsymbol{\Omega}^{\log}$	$\boldsymbol{\Omega}^{\log}$	$\mathbf{0}$

The logarithmic spin $\boldsymbol{\Omega}^{\log}$ was introduced recently in order that

$$\mathbf{D} = (\ln\mathbf{V})^{\cdot} - \boldsymbol{\Omega}^{\log}\ln\mathbf{V} + (\ln\mathbf{V})\boldsymbol{\Omega}^{\log}, \tag{25}$$

where $(\ln\mathbf{V})^{\cdot}$ denotes the material time derivative of the Eulerian logarithmic strain tensor $\ln\mathbf{V}$. With the logarithmic spin $\boldsymbol{\Omega}^{\log}$, the logarithmic rate of any Eulerian symmetric tensor, say $\boldsymbol{\tau}$, is defined by

$$\overset{\circ}{\boldsymbol{\tau}}{}^{\log} := \dot{\boldsymbol{\tau}} - \boldsymbol{\Omega}^{\log}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\Omega}^{\log}. \tag{26}$$

In particular, $\overset{\circ}{\boldsymbol{\tau}}{}^{\log}$ is referred as the stress rate of Xiao–Bruhns–Meyers, or simply as the Xiao–Bruhns–Meyers rate (see, e.g., Xiao et al., 1997a,b; Bruhns et al., 1999; Liu and Hong, 1999). Furthermore, Xiao et al. (1999) have proved that the basic constitutive equation (18) with its $\overset{\circ}{\boldsymbol{\tau}}$ replaced by the above logarithmic rate for $\boldsymbol{\tau}$ is the unique integrable hypoelastic equation of grade zero, leading exactly to the following elastic equation:

$$\boldsymbol{\tau} = \lambda(\ln(\det\mathbf{V})) + 2G\ln\mathbf{V}. \tag{27}$$

For the considered elastic–perfectly plastic models, Prager (1960) has proposed a yielding-stationarity criterion, which asserts that for a consistent flow model the vanishing of the stress rate implies the stationarity of yield function. It means that the stress rate must be a corotational rate. See, e.g., the explanation made by Lee (1983) for the Jaumann rate, and the proof made by Xiao et al. (2000) for the uniqueness of the logarithmic rate. Through the study to be conducted in Sections 3–5 it will be clear that the employment of the non-corotational stress rates in the constitutive equations leads to the coupling of deviatoric and volumetric parts of the constitutive equations as well as the loss of linearity in an augmented stress formulation. The two advantages of *uncoupling* and *linearity* strongly support the use of corotational stress rates in the constitutive equations, especially, for the purpose of numerical integrations.

3. A non-linear representation

Upon substituting Eq. (23) into Eq. (18) we obtain

$$\dot{\boldsymbol{\tau}} = \mathbf{B}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{B}^{\mathrm{T}} + 2G\mathbf{D}^{e} + \lambda(\operatorname{tr}\mathbf{D})\mathbf{I}_3. \tag{28}$$

In order to derive the objective stress rate for $\mathbf{s}$, denoted by $\overset{\diamond}{\mathbf{s}}$, such that

$$\overset{\diamond}{\mathbf{s}} = 2G\mathbf{D}'^{e} \tag{29}$$

is precisely a hypoelastic constitutive equation for $\mathbf{s}$, let us first take the trace of Eq. (28) to obtain

$$\operatorname{tr}\dot{\boldsymbol{\tau}} = 2\mathbf{B}\cdot\boldsymbol{\tau} + 3K\operatorname{tr}\mathbf{D}, \tag{30}$$

where $3K = 2G + 3\lambda = 2G(1+v)/(1-2v)$, and v is Poisson's ratio, and then decompose Eq. (28) into the deviatoric and volumetric parts:

$$\dot{\mathbf{s}} + \frac{1}{3}(\operatorname{tr}\dot{\boldsymbol{\tau}})\mathbf{I}_3 = \mathbf{B}\left(\mathbf{s} + \frac{1}{3}(\operatorname{tr}\boldsymbol{\tau})\mathbf{I}_3\right) + \left(\mathbf{s} + \frac{1}{3}(\operatorname{tr}\boldsymbol{\tau})\mathbf{I}_3\right)\mathbf{B}^{\mathrm{T}} + 2G\mathbf{D}'^{\mathrm{e}} + K(\operatorname{tr}\mathbf{D})\mathbf{I}_3, \tag{31}$$

where $\operatorname{tr}\mathbf{D}^{\mathrm{e}} = \operatorname{tr}\mathbf{D}$ was used due to the assumption of $\operatorname{tr}\mathbf{D}^{\mathrm{p}} = 0$, and $\mathbf{D}'^{\mathrm{e}}$ is the deviator of $\mathbf{D}^{\mathrm{e}}$. Thus, substituting Eq. (30) into Eq. (31) gives

$$\dot{\mathbf{s}} - \mathbf{B}\mathbf{s} - \mathbf{s}\mathbf{B}^{\mathrm{T}} - \frac{1}{3}(\operatorname{tr}\boldsymbol{\tau})(\mathbf{B} + \mathbf{B}^{\mathrm{T}}) + \frac{2}{3}(\mathbf{B}\cdot\boldsymbol{\tau})\mathbf{I}_3 = 2G\mathbf{D}'^{e}. \tag{32}$$

This relation promotes us to define

$$\overset{\diamond}{\mathbf{s}} := \dot{\mathbf{s}} - 2[\mathbf{B}\mathbf{s}] - \frac{1}{3}(\operatorname{tr}\boldsymbol{\tau})(\mathbf{B} + \mathbf{B}^{\mathrm{T}}) + \frac{2}{3}(\mathbf{B}\cdot\boldsymbol{\tau})\mathbf{I}_3 \tag{33}$$

as the objective stress rate for $\mathbf{s}$. Note that the replacement of $\overset{\diamond}{\mathbf{s}}$ in Eq. (29) by $\overset{\circ}{\mathbf{s}} = \dot{\mathbf{s}} - 2[\mathbf{B}\mathbf{s}]$ is incorrect, unless $\mathbf{B}$ is a skew-symmetric tensor, for which case $\mathbf{B} + \mathbf{B}^{\mathrm{T}} = \mathbf{0}$ and $\mathbf{B}\cdot\boldsymbol{\tau} = 0$, and hence $\overset{\diamond}{\mathbf{s}} = \overset{\circ}{\mathbf{s}}$.

Now we decompose $\mathbf{B}$ into its symmetric part $[\mathbf{B}] := (\mathbf{B} + \mathbf{B}^{\mathrm{T}})/2$ and its skew-symmetric part $\langle\mathbf{B}\rangle := (\mathbf{B} - \mathbf{B}^{\mathrm{T}})/2$, i.e.,

$$\mathbf{B} = [\mathbf{B}] + \langle\mathbf{B}\rangle. \tag{34}$$

As a consequence, Eq. (33) becomes

$$\overset{\diamond}{\mathbf{s}} = \dot{\mathbf{s}} - (\langle\mathbf{B}\rangle + [\mathbf{B}])\mathbf{s} + \mathbf{s}(\langle\mathbf{B}\rangle - [\mathbf{B}]) - \frac{2}{3}(\operatorname{tr}\boldsymbol{\tau})[\mathbf{B}] + \frac{2}{3}([\mathbf{B}]\cdot\mathbf{s})\mathbf{I}_3 + \frac{2}{9}(\operatorname{tr}[\mathbf{B}])(\operatorname{tr}\boldsymbol{\tau})\mathbf{I}_3. \tag{35}$$

If $[\mathbf{B}] = \mathbf{0}$ the above stress rate reduces to the usual corotational stress rate $\overset{\circ}{\mathbf{s}}$ for $\mathbf{s}$, and Eqs. (29) and (30) are uncoupled due to $\mathbf{B}\cdot\boldsymbol{\tau} = 0$. However, if $[\mathbf{B}] \neq \mathbf{0}$, Eqs. (29) and (30) are coupled together. Table 1 reveals that only the stress rates of Jaumann, Green–Naghdi, Sowerby–Chu and Xiao–Bruhns–Meyers result in uncoupled deviatoric and volumetric constitutive equations, and the other six non-corotational stress rates lead to the more complex deviatoric and volumetric coupled constitutive equations.

To proceed, let us further analyze the constitutive model (17)–(23). Substituting Eqs. (29), (19) and (35) into the deviatoric part of Eq. (17), we obtain

$$\dot{\mathbf{s}} - \langle\mathbf{B}\rangle\mathbf{s} + \mathbf{s}\langle\mathbf{B}\rangle + \frac{\dot{\lambda}}{\gamma_{\mathrm{y}}}\mathbf{s} = 2G\mathbf{P}, \tag{36}$$

where

$$\gamma_{\mathrm{y}} := \frac{\tau_{\mathrm{y}}^0}{G} \tag{37}$$

is the shear yield strain, and

$$\mathbf{P} := \mathbf{D}' + \frac{1}{2G}([\mathbf{B}]\mathbf{s} + \mathbf{s}[\mathbf{B}]) + \frac{1}{3G}(\operatorname{tr}\boldsymbol{\tau})[\mathbf{B}] - \frac{1}{3G}([\mathbf{B}]\cdot\mathbf{s})\mathbf{I}_3 - \frac{1}{9G}(\operatorname{tr}[\mathbf{B}])(\operatorname{tr}\boldsymbol{\tau})\mathbf{I}_3. \tag{38}$$

The inner product of $\mathbf{s}$ with Eq. (36) is

$$\mathbf{s}\cdot\dot{\mathbf{s}} + \frac{\dot{\lambda}}{\gamma_{\mathrm{y}}}\mathbf{s}\cdot\mathbf{s} = 2G\mathbf{s}\cdot\mathbf{P}, \tag{39}$$

such that

$$\|\mathbf{s}\| = \sqrt{2}\tau_{\mathrm{y}}^0 \Rightarrow \tau_{\mathrm{y}}^0\dot{\lambda} = \mathbf{s}\cdot\mathbf{P}. \tag{40}$$

Recalling $\tau_{\mathrm{y}}^0 > 0$, we have

$$\|\mathbf{s}\| = \sqrt{2}\tau_{\mathrm{y}}^0 \Rightarrow \{\mathbf{s}\cdot\mathbf{P} > 0 \iff \dot{\lambda} > 0\}, \tag{41}$$

and hence,

$$\{\|\mathbf{s}\| = \sqrt{2}\tau_{\mathrm{y}}^0 \quad \text{and} \quad \mathbf{s}\cdot\mathbf{P} > 0\} \Rightarrow \dot{\lambda} > 0. \tag{42}$$

On the other hand, if $\dot{\lambda} > 0$, Eq. (22) assures $\|\mathbf{s}\| = \sqrt{2}\tau_{\mathrm{y}}^0$, which together with Eq. (41) assert that

$$\dot{\lambda} > 0 \Rightarrow \{\|\mathbf{s}\| = \sqrt{2}\tau_{\mathrm{y}}^0 \quad \text{and} \quad \mathbf{s}\cdot\mathbf{P} > 0\}. \tag{43}$$

Statements (42) and (43) tell us that the yield condition $\|\mathbf{s}\| = \sqrt{2}\tau_y^0$ and the loading condition $\mathbf{s}\cdot\mathbf{P} > 0$ are sufficient and necessary for plastic irreversibility $\dot{\lambda} > 0$. In view of Eqs. (20), (21) and (40), the two statements are logically equivalent to the following criteria:

$$\dot{\lambda} = \begin{cases} \frac{1}{\tau_y^0}\mathbf{s}\cdot\mathbf{P} > 0 & \text{if } \|\mathbf{s}\| = \sqrt{2}\tau_y^0 \quad \text{and} \quad \mathbf{s}\cdot\mathbf{P} > 0, \\ 0 & \text{if } \|\mathbf{s}\| < \sqrt{2}\tau_y^0 \quad \text{or} \quad \mathbf{s}\cdot\mathbf{P} \leqslant 0. \end{cases} \tag{44}$$

From Eqs. (36) and (44) follows a two-phase non-linear system of differential equations:

$$\dot{\mathbf{s}} - \langle\mathbf{B}\rangle\mathbf{s} + \mathbf{s}\langle\mathbf{B}\rangle = \begin{cases} -\frac{\mathbf{s}\cdot\mathbf{P}}{\tau_y^0\gamma_y}\mathbf{s} + 2G\mathbf{P} & \text{if } \|\mathbf{s}\| = \sqrt{2}\tau_y^0 \quad \text{and} \quad \mathbf{s}\cdot\mathbf{P} > 0, \\ 2G\mathbf{P} & \text{if } \|\mathbf{s}\| < \sqrt{2}\tau_y^0 \quad \text{or} \quad \mathbf{s}\cdot\mathbf{P} \leqslant 0. \end{cases} \tag{45}$$

According to criteria (44) and the complementary trios (20)–(22) and further to the two-phase system (45), the model of elastoplasticity has precisely two phases: the on phase in which $\dot{\lambda} > 0$ and $\|\mathbf{s}\| = \sqrt{2}\tau_y^0$ and the off phase in which $\dot{\lambda} = 0$ and $\|\mathbf{s}\| \leqslant \sqrt{2}\tau_y^0$. In the on phase the plasticity mechanism is on so that the model exhibits elastoplastic behavior, which is irreversible, while in the off phase the plasticity mechanism is off so that the model responds elastically and reversibly. Thus, Eq. (44) is called the on-off switching criteria for the mechanism of plasticity.

Eq. (45) is a *cubic degree non-linear* representation of the constitutive model upon noting that $\mathbf{P}$ is a linear function of $\boldsymbol{\tau}$. However, if $[\mathbf{B}] = \mathbf{0}$, $\mathbf{P}$ reduces to $\mathbf{D}'$, and then Eq. (45) reduces to *quadratic non-linear* equation in the plastic phase. No matter which stress rate is adopted, the resulting constitutive equations are non-linear in nature.

4. A six-dimensional Lie type representation

From Eq. (38) it is obvious that $\operatorname{tr}\mathbf{P} = 0$. Due to the zero traces of the deviatoric tensors $\mathbf{s}$ and $\mathbf{P}$, i.e.,

$$s^{33} = -s^{11} - s^{22}, \quad P_{33} = -P_{11} - P_{22}, \tag{46}$$

the dimensions of Eq. (45) can be reduced to five. Let us introduce the integrating factor

$$X^0 := \exp\left(\frac{\lambda}{\gamma_y}\right), \tag{47}$$

and the following six-dimensional augmented stress vector:

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 \\ X^4 \\ X^5 \\ X^0 \end{bmatrix} := \frac{X^0}{\tau_y^0} \begin{bmatrix} \beta_1 s^{11} + \beta_2 s^{22} \\ \beta_3 s^{11} + \beta_4 s^{22} \\ s^{23} \\ s^{13} \\ s^{12} \\ \tau_y^0 \end{bmatrix}. \tag{48}$$

Then the on-off switching criteria turn out to be

$$\dot{X}^0 = \begin{cases} \mathbf{A}_s^0\mathbf{X}^s > 0 & \text{if } \mathbf{X}^T\mathbf{g}\mathbf{X} = 0 \quad \text{and} \quad \frac{d}{dt}\left[(\mathbf{X}^s)^T\mathbf{g}_{ss}\mathbf{X}^s\right] > 0, \\ 0 & \text{if } \mathbf{X}^T\mathbf{g}\mathbf{X} < 0 \quad \text{or} \quad \frac{d}{dt}\left[(\mathbf{X}^s)^T\mathbf{g}_{ss}\mathbf{X}^s\right] \leqslant 0, \end{cases} \tag{49}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}_{ss} & \mathbf{g}_{s0} \\ \mathbf{g}_{0s} & g_{00} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_5 & \mathbf{0}_{5\times 1} \\ \mathbf{0}_{1\times 5} & -1 \end{bmatrix}, \tag{50}$$

$$(\mathbf{A}_s^0)^{\mathrm{T}} = \mathbf{A}_0^s = \begin{bmatrix} A_0^1 \\ A_0^2 \\ A_0^3 \\ A_0^4 \\ A_0^5 \end{bmatrix} := \frac{2}{\gamma_y} \begin{bmatrix} \beta_1 P_{11} + \beta_2 P_{22} \\ \beta_3 P_{11} + \beta_4 P_{22} \\ P_{23} \\ P_{13} \\ P_{12} \end{bmatrix}. \tag{51}$$

Therefore, similar to the work by Hong and Liu (1999), we can put Eq. (45) to the following quasilinear system:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}, \tag{52}$$

where

$$\mathbf{A} = \begin{cases} \begin{bmatrix} \mathbf{A}_s^s & \mathbf{A}_0^s \\ \mathbf{A}_s^0 & 0 \end{bmatrix} & \text{if } \mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{T}}\mathbf{g}_{ss}\mathbf{X}^s] > 0, \\ \begin{bmatrix} \mathbf{A}_s^s & \mathbf{A}_0^s \\ \mathbf{0}_{1\times 5} & 0 \end{bmatrix} & \text{if } \mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} < 0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{X}^s)^{\mathrm{T}}\mathbf{g}_{ss}\mathbf{X}^s] \leqslant 0, \end{cases} \tag{53}$$

in which

$$\mathbf{A}_s^s := \begin{bmatrix} 0 & 0 & 2\beta_2\langle B\rangle_{23} & 2\beta_1\langle B\rangle_{13} & 2(\beta_1-\beta_2)\langle B\rangle_{12} \\ & 0 & 2\beta_4\langle B\rangle_{23} & 2\beta_3\langle B\rangle_{13} & 2(\beta_3-\beta_4)\langle B\rangle_{12} \\ & & 0 & -\langle B\rangle_{12} & -\langle B\rangle_{13} \\ & \text{skew-sym.} & & 0 & -\langle B\rangle_{23} \\ & & & & 0 \end{bmatrix} \tag{54}$$

is skew-symmetric, i.e., $(\mathbf{A}_s^s)^{\mathrm{T}} = -\mathbf{A}_s^s$.

Note that Eq. (52) is a (5 + 1)-dimensional Lie algebra type representation of the constitutive model (17)–(22), in which **X** and **A** are the augmented stress vector and the control tensor, respectively. If Eq. (52) is viewed as a matrix representation, the $(5+1)\times 1$ matrix **X** contains the contravariant components of the augmented stress vector **X**, and the $(5+1)\times(5+1)$ matrix **A** contains the mixed components of the control tensor **A**.

5. $PSO_o(5,1)$ symmetry in the plastic phase

Because the elastic phase equations are rather simple, hereafter, we concentrate on the plastic phase to bring out internal symmetry inherent in the model in the plastic phase. Denote by I_{on} an open, maximal, continuous time interval during which the mechanism of plasticity is on exclusively. From Eqs. $(53)_1$ and (50) it is easy to verify that the system matrix **A** in the plastic phase satisfies

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}. \tag{55}$$

Hence, the corresponding transformation **G**, generating from the solution of

$$\dot{\mathbf{G}}(t) = \mathbf{A}(t)\mathbf{G}(t), \tag{56}$$

$$\mathbf{G}(0) = \mathbf{I}_6, \tag{57}$$

satisfies

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{58}$$

$$\det \mathbf{G} = 1, \tag{59}$$

$$G_0^0 \geqslant 1. \tag{60}$$

Thereby the plastic phase control tensor $\mathbf{A}$ is an element of the real Lie algebra $so(5,1)$ and generates the plastic phase transformation $\mathbf{G}$, which is thus an element of the proper orthochronous Lorentz group $SO_o(5,1)$; refer Liu (2001b) for a more detailed discussion. The function $\mathbf{G}(t)$ of time $t \in \mathrm{I_{on}}$ may be viewed as a connected path of the Lorentz group and the algebraic and topological properties of the proper orthochronous Lorentz group are shared by the constitutive model in the plastic phase.

We solve Eq. (58) for the inverse

$$\mathbf{G}^{-1} = \mathbf{g}\mathbf{G}^{\mathrm{T}}\mathbf{g} \tag{61}$$

and partition $\mathbf{G}$ as

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_{\mathrm{s}}^{\mathrm{s}} & \mathbf{G}_0^{\mathrm{s}} \\ \mathbf{G}_{\mathrm{s}}^0 & G_0^0 \end{bmatrix}, \tag{62}$$

where $\mathbf{G}_{\mathrm{s}}^{\mathrm{s}}$, $\mathbf{G}_0^{\mathrm{s}}$ and $\mathbf{G}_{\mathrm{s}}^0$ are of order 5×5, 5×1 and 1×5, respectively. Thus, we obtain the following augmented stress transition equation:

$$\begin{bmatrix} \mathbf{X}^{\mathrm{s}}(t) \\ X^0(t) \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{\mathrm{s}}^{\mathrm{s}}(t)(\mathbf{G}_{\mathrm{s}}^{\mathrm{s}})^{\mathrm{T}}(t_1) - \mathbf{G}_0^{\mathrm{s}}(t)(\mathbf{G}_0^{\mathrm{s}})^{\mathrm{T}}(t_1) & \mathbf{G}_0^{\mathrm{s}}(t)G_0^0(t_1) - \mathbf{G}_{\mathrm{s}}^{\mathrm{s}}(t)\mathbf{G}_{\mathrm{s}}^0(t_1) \\ \mathbf{G}_{\mathrm{s}}^0(t)(\mathbf{G}_{\mathrm{s}}^{\mathrm{s}})^{\mathrm{T}}(t_1) - G_0^0(t)(\mathbf{G}_0^{\mathrm{s}})^{\mathrm{T}}(t_1) & G_0^0(t)G_0^0(t_1) - \mathbf{G}_{\mathrm{s}}^0(t)\mathbf{G}_{\mathrm{s}}^0(t_1) \end{bmatrix} \begin{bmatrix} \mathbf{X}^{\mathrm{s}}(t_1) \\ X^0(t_1) \end{bmatrix}, \tag{63}$$

which is valid for the plastic phase.

Once the augmented stress vector $\mathbf{X}(t)$ is obtained, from Eq. (48) the deviatoric stress $\mathbf{s}(t)$ can be determined as follows:

$$\begin{bmatrix} s^{11} \\ s^{22} \\ s^{23} \\ s^{13} \\ s^{12} \end{bmatrix} = \begin{bmatrix} \beta_4 & -\beta_2 & \mathbf{0}_{2\times 3} \\ -\beta_3 & \beta_1 & \\ \mathbf{0}_{3\times 2} & & \frac{\sqrt{3}}{2}\mathbf{I}_3 \end{bmatrix} \frac{2\tau_{\mathrm{y}}^0}{\sqrt{3}X^0}\mathbf{X}^{\mathrm{s}}. \tag{64}$$

By this and the plastic phase transition formula (63) one can map $\mathbf{s}(t_1)$ to the current $\mathbf{s}(t)$.

Here, we emphasize that $[\mathbf{B}] = \mathbf{0}$ and $\langle \mathbf{B} \rangle = \mathbf{B}$ for the use of corotational stress rates in the constitutive equations, and hence $\mathbf{P}$ reduces to $\mathbf{D}'$ by Eq. (38). Therefore, $(\mathbf{A}_{\mathrm{s}}^0)^{\mathrm{T}} = \mathbf{A}_0^{\mathrm{s}}$ defined in Eq. (51) and $\mathbf{A}_{\mathrm{s}}^{\mathrm{s}}$ defined in Eq. (54) are both functions of t through the deformation history. Accordingly, the $\mathbf{A}$ defined in Eq. (53) is a state matrix depending only on time t, which means that Eq. (52) is a *linear* ODE system. From the Prager yielding-stationarity criterion which as demonstrated by Xiao et al. (2000) means that the stress rates must be corotational, and the above discussions we can exactly linearize the finite strain elastoplastic models in the augmented stress space. This version presents an extremely simple framework with obvious advantage for numerical integrations.

6. Quaternionic two-component spinor representation

In order to give a more economic lower dimensional quaternionic two-component spinor representation of the augmented stress $\mathbf{X}$ defined in Eq. (48), which in plastic phase subjects to the following constraint:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0. \tag{65}$$

Let us consider the 2×2 quaternionic Hermitian matrix $\mathbf{H}$, which can be written as

$$\mathbf{H} = \begin{bmatrix} x^0 + x^5 & x^1 - x^2\mathbf{i}_2 - x^3\mathbf{i}_3 - x^4\mathbf{i}_4 \\ x^1 + x^2\mathbf{i}_2 + x^3\mathbf{i}_3 + x^4\mathbf{i}_4 & x^0 - x^5 \end{bmatrix}, \tag{66}$$

where $\mathbf{i}_1 = 1$, $\mathbf{i}_2$, $\mathbf{i}_3$ and $\mathbf{i}_4$ are four distinct bases of quaternions, and x^0, x^1, x^2, x^3, x^4 and x^5 are six real numbers. The minus of the determinant of $\mathbf{H}$ is just the Minkowski separation $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 - (x^0)^2$. Some properties of quaternion algebra, denoted by $\mathbb{H}$, are summarized in Appendix A.

The group $SL(2,\mathbb{H})$ is the set of all quaternionic 2×2 matrices $\mathbf{U}$ with unit determinant. Elements of $SL(2,\mathbb{H})$ are often called quaternion spin transformations. Hence $\mathbf{U} \in SL(2,\mathbb{H})$ is a spin transformation. Since $\mathbf{H}$ is quaternionic Hermitian, it is obvious that $\mathbf{U}\mathbf{H}\bar{\mathbf{U}}^{\mathrm{T}}$ is also a 2×2 quaternionic Hermitian matrix. This led us to write

$$\widehat{\mathbf{H}} = \mathbf{U}\mathbf{H}\overline{\mathbf{U}}^{\mathrm{T}}, \tag{67}$$

where the bar over $\mathbf{U}$ stands for its quaternion conjugate. Taking the determinants of both sides and using $\det\mathbf{U} = \det\overline{\mathbf{U}}^{\mathrm{T}} = 1$, one readily obtains $(\hat{x}^1)^2 + (\hat{x}^2)^2 + (\hat{x}^3)^2 + (\hat{x}^4)^2 + (\hat{x}^5)^2 - (\hat{x}^0)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 - (x^0)^2$, ensuring the Minkowski separation of the $(5+1)$-vector $\mathbf{X} = (x^1, x^2, x^3, x^4, x^5, x^0)$ is preserved by the spin transformation $\mathbf{U} : \mathbf{H} \mapsto \widehat{\mathbf{H}}$. Indeed, the transformation $\mathbf{U} : \mathbf{H} \mapsto \widehat{\mathbf{H}}$ induces a proper orthochronous Lorentz transformation $\mathbf{G} : \mathbf{X} \mapsto \widehat{\mathbf{X}}$, which is an element of $SO_o(5,1)$.

Here we are concerned with the quaternionic two-component spinor space (see, e.g., Naber, 1997) and the dynamical systems on this space. We return to the matrix $\mathbf{H}$ defined in Eq. (66) and remark that if the vector $\mathbf{X} \in \mathbb{M}^{5+1}$ is null, i.e., $\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = 0$, then $\mathbf{H}$ may be written as the dyadic product of a quaternion two-dimensional vector and its conjugate transpose:

$$\mathbf{H} = \begin{bmatrix} x^0 + x^5 & x^1 - \mathbf{i}\mathbf{x}^{\mathrm{s}} \\ x^1 + \mathbf{i}\mathbf{x}^{\mathrm{s}} & x^0 - x^5 \end{bmatrix} = 2\begin{bmatrix} \boldsymbol{\alpha}^1\bar{\boldsymbol{\alpha}}^1 & \boldsymbol{\alpha}^1\bar{\boldsymbol{\alpha}}^2 \\ \boldsymbol{\alpha}^2\bar{\boldsymbol{\alpha}}^1 & \boldsymbol{\alpha}^2\bar{\boldsymbol{\alpha}}^2 \end{bmatrix} = 2\boldsymbol{\alpha}\bar{\boldsymbol{\alpha}}^{\mathrm{T}}, \tag{68}$$

where we used a new notation $\mathbf{x} = x^1 + \mathbf{i}\mathbf{x}^{\mathrm{s}}$ to denote the quaternion (e.g., Liu (2002)), of which x^1 is the scalar part of $\mathbf{x}$ and $\mathbf{x}^{\mathrm{s}}$ having three components is the vectorial part of $\mathbf{x}$. We are therefore led to consider a quaternion two-dimensional vector space $\mathbb{H}^2$ with elements $\boldsymbol{\alpha}$, on which $SL(2,\mathbb{H})$ left acts. This is a *quaternion spin-space* and the elements are *quaternion spinors*. However, how to realize explicitly the above spinor map from $SL(2,\mathbb{H})$ onto $SO_o(5,1)$, and how to construct explicitly the transform between their Lie algebras $sl(2,\mathbb{H})$ and $so(5,1)$ are still pending in the literature. This realizations require a lot of algebraic constructions based on the quaternion algebra as shown in Fig. 1. We thus relegate those detailed derivations in Appendices B and C.

From Eq. (68) it follows that

$$\begin{aligned} &x^0 = \|\boldsymbol{\alpha}\|^2 = \|\boldsymbol{\alpha}^1\|^2 + \|\boldsymbol{\alpha}^2\|^2, \quad x^1 = 2\,\mathrm{Sca}(\boldsymbol{\alpha}^1\bar{\boldsymbol{\alpha}}^2), \\ &\mathbf{x}^{\mathrm{s}} = 2\,\mathrm{Vec}(\boldsymbol{\alpha}^2\bar{\boldsymbol{\alpha}}^1), \quad x^5 = \|\boldsymbol{\alpha}^1\|^2 - \|\boldsymbol{\alpha}^2\|^2, \end{aligned} \tag{69}$$

where Sca and Vec denote the scalar and vector parts of quaternion, respectively. It deserves to note that the two $\pm\boldsymbol{\alpha}$ lead to the same $\mathbf{X}$, and the *two quaternion components* $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$ suffice to determine the *six real components* $x^1, \ldots, x^5, x^0$ as shown in Eq. (69), and that the map $SL(2,\mathbb{H}) \to SO_o(5,1)$ as shown in Eq. (B.25) is a two-to-one surjective covering. With this advantage we may identify $\mathbf{X} = (\mathbf{X}^{\mathrm{s}}, X^0) = (x^1, \mathbf{x}^{\mathrm{s}}, x^5, x^0)$ and consider the following equations system for quaternion spinor:

$$\dot{\boldsymbol{\alpha}} = \mathbf{Q}\boldsymbol{\alpha}, \tag{70}$$

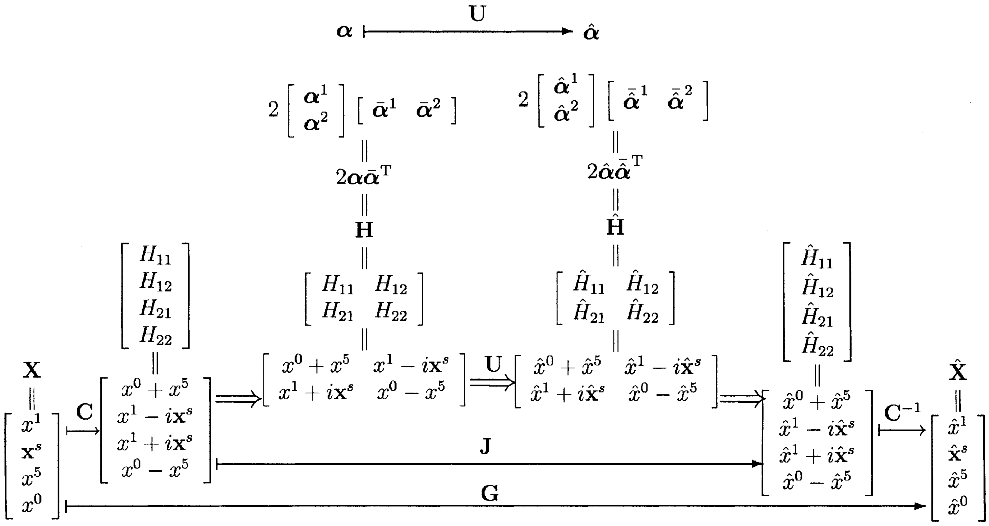

Fig. 1. The algebraic procedure for constructing the spinor map from $SL(2,\mathbb{H})$ onto $SO_o(5,1)$.

where $\mathbf{Q}$ as defined in Eq. (C.37) is a 2×2 quaternion matrix determined uniquely by $\mathbf{A}$. If we can solve Eq. (70) instead of Eq. (52), it naturally gives $\mathbf{X}$ by the above correspondence (69).

7. $SU^*(4)$ group representation

Instead of $sl(2,\mathbb{H})$ we now turn our attention to the 4×4 complex matrix representation of $so(5,1)$. For this purpose we first associate the bases set $\mathbf{i}_1 = 1, \mathbf{i}_2, \mathbf{i}_3, \mathbf{i}_4$ of quaternion with the following 2×2 matrices:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \right\}, \tag{71}$$

denoted respectively by $\mathbf{I}_2$ and $\boldsymbol{\rho}_k$, $k = 2, 3, 4$. The latter three matrices are obtained from the Pauli matrices by multiplying by 'i'. Then, for any quaternion $\mathbf{x} = \sum_{k=1}^{4} x^k \mathbf{i}_k$ with x^1, x^2, x^3, x^4 real numbers we may let $\phi(\mathbf{x})$ be the 2×2 matrix defined by

$$\phi(\mathbf{x}) = x^1 \mathbf{I}_2 + \sum_{k=2}^{4} x^k \boldsymbol{\rho}_k = \begin{bmatrix} x^1 + \mathrm{i}x^2 & x^3 + \mathrm{i}x^4 \\ -x^3 + \mathrm{i}x^4 & x^1 - \mathrm{i}x^2 \end{bmatrix}. \tag{72}$$

The above ϕ provides us an isomorphic mapping of quaternions onto 2×2 matrices. Now we consider the following 4×4 complex matrix:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix}, \tag{73}$$

each sub-matrix $\mathbf{M}_{jk}$, $j, k = 1, 2$, is obtained by the above isomorphic mapping of the quaternions $\mathbf{o}$, $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ in Eq. (C.37), that is,

$$\mathbf{M} = \begin{bmatrix} r_1 + \mathrm{i}r_2 & r_3 + \mathrm{i}r_4 & q_1 + \mathrm{i}q_2 & q_3 + \mathrm{i}q_4 \\ -r_3 + \mathrm{i}r_4 & r_1 - \mathrm{i}r_2 & -q_3 + \mathrm{i}q_4 & q_1 - \mathrm{i}q_2 \\ p_1 + \mathrm{i}p_2 & p_3 + \mathrm{i}p_4 & o_1 + \mathrm{i}o_2 & o_3 + \mathrm{i}o_4 \\ -p_3 + \mathrm{i}p_4 & p_1 - \mathrm{i}p_2 & -o_3 + \mathrm{i}o_4 & o_1 - \mathrm{i}o_2 \end{bmatrix}. \tag{74}$$

We can prove that such $\mathbf{M}$, satisfying

$$\mathbf{J}\mathbf{M}^* = \mathbf{M}\mathbf{J}, \tag{75}$$

$$\operatorname{tr}\mathbf{M} = 0, \tag{76}$$

is an element of $su^*(4)$. The latter equation is due to $\mathrm{Sca}(\mathbf{o}+\mathbf{r}) = 0$. Here, $*$ denotes the complex conjugate and $\mathbf{J}$ is defined by

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{77}$$

Thus, by means of Eqs. (C.37) and (74) we have established the isomorphic mapping formula between $sl(2,\mathbb{H})$ and $su^*(4)$, the latter of which generates the group $SU^*(4)$.

8. Group preserving schemes

8.1. Numerical scheme based on $SO_o(5,1)$

The simplest scheme for Eq. (52) is a time-centered Euler scheme (see, e.g., Liu, 2001b):

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \tau\mathbf{A}(\mathbf{X}_{n+1} + \mathbf{X}_n), \tag{78}$$

where $\mathbf{X}_n$ denotes the numerical value of $\mathbf{X}$ at the discrete time step t_n, that is, $\mathbf{X}_n = \mathbf{X}(t_n)$, and τ is one half of the time increment, i.e., $\tau := \Delta t/2 = (t_{n+1} - t_n)/2$. Using the Cayley transform we have

$$\mathbf{X}_{n+1} = \mathrm{Cay}(\tau\mathbf{A})\mathbf{X}_n := (\mathbf{I}_6 - \tau\mathbf{A})^{-1}(\mathbf{I}_6 + \tau\mathbf{A})\mathbf{X}_n = [\mathbf{I}_6 + 2\tau(\mathbf{I}_6 - \tau\mathbf{A})^{-1}\mathbf{A}]\mathbf{X}_n. \tag{79}$$

It is easy to check that this transform preserves the properties (58)–(60) of the proper orthochronous Lorentz group, i.e., $\mathrm{Cay}(\tau\mathbf{A}) \in SO_o(5,1)$. Through some derivations $\mathrm{Cay}(\tau\mathbf{A})$ was found to be (see, e.g., Hong and Liu, 1999a; Liu, 2001b)

$$\mathrm{Cay}(\tau\mathbf{A}) = \begin{bmatrix} \mathbf{I}_5 + 2c\tau^3\boldsymbol{\eta}\mathbf{A}_0^{\mathrm{s}}\mathbf{A}_{\mathrm{s}}^0\boldsymbol{\eta}\mathbf{A}_{\mathrm{s}}^{\mathrm{s}} + 2\tau\boldsymbol{\eta}\mathbf{A}_{\mathrm{s}}^{\mathrm{s}} + 2c\tau^2\boldsymbol{\eta}\mathbf{A}_0^{\mathrm{s}}\mathbf{A}_{\mathrm{s}}^0 & 2c\tau^3\boldsymbol{\eta}\mathbf{A}_0^{\mathrm{s}}\mathbf{A}_{\mathrm{s}}^0\boldsymbol{\eta}\mathbf{A}_0^{\mathrm{s}} + 2\tau\boldsymbol{\eta}\mathbf{A}_0^{\mathrm{s}} \\ 2c\tau^2\mathbf{A}_{\mathrm{s}}^0\boldsymbol{\eta}\mathbf{A}_{\mathrm{s}}^{\mathrm{s}} + 2c\tau\mathbf{A}_{\mathrm{s}}^0 & 1 + 2c\tau^2\mathbf{A}_{\mathrm{s}}^0\boldsymbol{\eta}\mathbf{A}_0^{\mathrm{s}} \end{bmatrix}, \tag{80}$$

where

$$\boldsymbol{\eta} := (\mathbf{I}_5 - \tau\mathbf{A}_{\mathrm{s}}^{\mathrm{s}})^{-1} = \mathbf{I}_5 + \rho_1\mathbf{A}_{\mathrm{s}}^{\mathrm{s}} + \rho_2(\mathbf{A}_{\mathrm{s}}^{\mathrm{s}})^2 + \rho_3(\mathbf{A}_{\mathrm{s}}^{\mathrm{s}})^3 + \rho_4(\mathbf{A}_{\mathrm{s}}^{\mathrm{s}})^4,$$

$$c := \frac{1}{1 - \tau^2\mathbf{A}_{\mathrm{s}}^0\boldsymbol{\eta}\mathbf{A}_0^{\mathrm{s}}},$$

in which

$$w := \sqrt{\langle B\rangle_{12}^2 + \langle B\rangle_{13}^2 + \langle B\rangle_{23}^2},$$

$$\rho_1 := \frac{\tau + 5w^2\tau^3}{1 + 5w^2\tau^2 + 4w^4\tau^4}, \quad \rho_2 := \frac{\tau^2 + 5w^2\tau^4}{1 + 5w^2\tau^2 + 4w^4\tau^4},$$

$$\rho_3 := \frac{\tau^3}{1 + 5w^2\tau^2 + 4w^4\tau^4}, \quad \rho_4 := \frac{\tau^4}{1 + 5w^2\tau^2 + 4w^4\tau^4}.$$

Once $\mathbf{X}_n$ is calculated at each time step, formula (64) gives the value of stress $\mathbf{s}_n$ at each time step. The above scheme together with the discretization of Eq. (30)

$$\operatorname{tr}\boldsymbol{\tau}_{n+1} = \operatorname{tr}\boldsymbol{\tau}_n + \Delta t(2\mathbf{B}_n \cdot \boldsymbol{\tau}_n + 3K\operatorname{tr}\mathbf{D}_n), \tag{81}$$

constitutes a numerical scheme to calculate the stress response $\boldsymbol{\tau}$.

A numerical algorithm is called a *group preserving scheme* if for every time increment the map from $\mathbf{X}_n$ to $\mathbf{X}_{n+1}$ preserves the group properties (58)–(60). Now let us investigate what $\operatorname{Cay}(\tau\mathbf{A}) \in SO_o(5,1)$ implies as a numerical scheme for the constitutive law of plasticity? From Eqs. (48), (58) and (79) it follows that

$$\mathbf{X}_{n+1}^{\mathrm{T}}\mathbf{g}\mathbf{X}_{n+1} = \mathbf{X}_n^{\mathrm{T}}\mathbf{g}\mathbf{X}_n = (X_{n+1}^0)^2\left[\frac{\|\mathbf{s}_{n+1}\|^2}{2(\tau_{\mathrm{y}}^0)^2} - 1\right] = (X_n^0)^2\left[\frac{\|\mathbf{s}_n\|^2}{2(\tau_{\mathrm{y}}^0)^2} - 1\right] = 0. \tag{82}$$

Because of $X_{n+1}^0 \geqslant X_n^0 > 0$, the equalities in Eq. (82) say nothing but for every time increment the points $\mathbf{s}_n$ and $\mathbf{s}_{n+1}$ are located on the yield hypersphere, i.e., $\|\mathbf{s}_{n+1}\| = \|\mathbf{s}_n\| = \sqrt{2}\tau_{\mathrm{y}}^0$. In other words, the consistency condition is fulfilled exactly for every time step in the plastic phase. Therefore, the new numerical scheme may be specifically called an exact *consistency scheme*. This is what the conventional schemes of computational plasticity desired and failed to achieve.

8.2. Numerical scheme based on SL(2,ℍ)

In order to develop the scheme based on the symmetry group $SL(2,\mathbb{H})$, we first need to know the relation of $\mathbf{Q}$ and $\mathbf{A}$. Comparing Eqs. (C.37) and (C.34) and with the $\mathbf{A}$ in Eq. $(53)_1$, we get

$$\mathbf{A}_{ss} = \begin{bmatrix} 0 & 2\beta_4\langle B\rangle_{23} & 2\beta_3\langle B\rangle_{13} \\ -2\beta_4\langle B\rangle_{23} & 0 & -\langle B\rangle_{12} \\ -2\beta_3\langle B\rangle_{13} & \langle B\rangle_{12} & 0 \end{bmatrix} \tag{83}$$

and thus the axial vector of $\mathbf{A}_{ss}$ is given by

$$\operatorname{axial}(\mathbf{A}_{ss}) = \begin{bmatrix} \langle B\rangle_{12} \\ 2\beta_3\langle B\rangle_{13} \\ -2\beta_4\langle B\rangle_{23} \end{bmatrix}. \tag{84}$$

The other quantities are given by

$$\mathbf{A}_{1s} = \begin{bmatrix} 0 \\ 2\beta_2\langle B\rangle_{23} \\ 2\beta_1\langle B\rangle_{13} \end{bmatrix}, \quad \mathbf{A}_{s5} = \begin{bmatrix} 2(\beta_3 - \beta_4)\langle B\rangle_{12} \\ -\langle B\rangle_{13} \\ -\langle B\rangle_{23} \end{bmatrix}, \quad \mathbf{A}_{s0} = \frac{2}{\gamma_{\mathrm{y}}}\begin{bmatrix} \beta_3 P_{11} + \beta_4 P_{22} \\ P_{23} \\ P_{13} \end{bmatrix},$$
$$A_{10} = \frac{2}{\gamma_{\mathrm{y}}}(\beta_1 P_{11} + \beta_2 P_{22}), \quad A_{50} = \frac{2}{\gamma_{\mathrm{y}}}P_{12}, \quad A_{15} = 2(\beta_1 - \beta_2)\langle B\rangle_{12}. \tag{85}$$

Substituting Eqs. (84) and (85) into Eq. (C.37) we obtain the four quaternion components of $\mathbf{Q}$ as follows:

$$\mathbf{r} = \frac{1}{\gamma_y}P_{12} + \mathrm{i}\begin{bmatrix} \frac{1}{2}\langle B\rangle_{12} \\ \beta_3\langle B\rangle_{13} + \beta_2\langle B\rangle_{23} \\ \beta_1\langle B\rangle_{13} - \beta_4\langle B\rangle_{23} \end{bmatrix},$$

$$\mathbf{q} = \frac{1}{\gamma_y}(\beta_1 P_{11} + \beta_2 P_{22}) - (\beta_1 - \beta_2)\langle B\rangle_{12} + \mathrm{i}\begin{bmatrix} (\beta_3 - \beta_4)\langle B\rangle_{12} - \frac{1}{\gamma_y}(\beta_3 P_{11} + \beta_4 P_{22}) \\ -\frac{1}{2}\langle B\rangle_{13} - \frac{1}{\gamma_y}P_{23} \\ -\frac{1}{2}\langle B\rangle_{23} - \frac{1}{\gamma_y}P_{13} \end{bmatrix},$$

$$\mathbf{p} = \frac{1}{\gamma_y}(\beta_1 P_{11} + \beta_2 P_{22}) + (\beta_1 - \beta_2)\langle B\rangle_{12} + \mathrm{i}\begin{bmatrix} (\beta_3 - \beta_4)\langle B\rangle_{12} + \frac{1}{\gamma_y}(\beta_3 P_{11} + \beta_4 P_{22}) \\ \frac{1}{\gamma_y}P_{23} - \frac{1}{2}\langle B\rangle_{13} \\ \frac{1}{\gamma_y}P_{13} - \frac{1}{2}\langle B\rangle_{23} \end{bmatrix}, \tag{86}$$

$$\mathbf{o} = -\frac{1}{\gamma_y}P_{12} + \mathrm{i}\begin{bmatrix} \frac{1}{2}\langle B\rangle_{12} \\ \beta_3\langle B\rangle_{13} - \beta_2\langle B\rangle_{23} \\ -\beta_1\langle B\rangle_{13} - \beta_4\langle B\rangle_{23} \end{bmatrix}.$$

As done in Eq. (79) we simply approximate the solution of Eq. (C.1) by the following Cayley transformation:

$$\mathbf{U} = \mathrm{Cay}(\tau\mathbf{Q}) := (\mathbf{I}_2 - \tau\mathbf{Q})^{-1}(\mathbf{I}_2 + \tau\mathbf{Q}) = \mathbf{I}_2 + 2\tau(\mathbf{I}_2 - \tau\mathbf{Q})^{-1}\mathbf{Q}. \tag{87}$$

Substituting Eq. (C.37) for $\mathbf{Q}$ into the above equation and through some manipulations, we get

$$\mathbf{U} = \mathrm{Cay}(\tau\mathbf{Q}) = \mathbf{I}_2 + 2\tau[(1 - \tau\mathbf{r})(1 - \tau\mathbf{o}) - \tau^2\mathbf{pq}]^{-1}\begin{bmatrix} (1 - \tau\mathbf{o})\mathbf{r} + \tau\mathbf{qp} & (1 - \tau\mathbf{o})\mathbf{q} + \tau\mathbf{qo} \\ (1 - \tau\mathbf{r})\mathbf{p} + \tau\mathbf{pr} & (1 - \tau\mathbf{r})\mathbf{o} + \tau\mathbf{pq} \end{bmatrix}. \tag{88}$$

The inverse in the above is calculated according to formula (A.8). Upon obtaining $\mathbf{U}$ we can compute $\mathbf{G}$ by Eqs. (B.27)–(B.42). This however needs a lot of algebraic calculations.

8.3. Numerical scheme based on $SU^*(4)$

In this section, let us mention the third type scheme, which is formulated according to the symmetry group $SU^*(4)$. Substituting Eq. (74) for $\mathbf{M}$ into

$$\mathrm{Cay}(\tau\mathbf{M}) := (\mathbf{I}_4 - \tau\mathbf{M})^{-1}(\mathbf{I}_4 + \tau\mathbf{M}) = \mathbf{I}_4 + 2\tau(\mathbf{I}_4 - \tau\mathbf{M})^{-1}\mathbf{M}, \tag{89}$$

we obtain the Cayley transformation of $SU^*(4)$. Converting this result by the isomorphomic mapping formula, similar to Eq. (74), from the 4×4 complex matrix to the four quaternions **a**, **b**, **c**, and **d** as arranged in Eq. (B.11), we thus obtain the corresponding $\mathbf{U} \in SL(2, \mathbb{H})$. Finally, by means of Eqs. (B.27)–(B.42) we obtain its corresponding $\mathbf{G}$. This scheme needing to calculate the inverse of the 4×4 complex matrix and also two transformations is more time consumption than the other two numerical schemes. However, it gives almost the same numerical results as that provided by the previous two numerical schemes.

8.4. Numerical results

In order to compare the effects of different objective stress rates on the model behavior let us consider the elastoplastic models by employing the ten objective stress rates listed in Table 1 under simple shear deformation, whose deformation gradient is

$$\mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \gamma \in [0, \infty), \tag{90}$$

where γ is the shear engineering strain. Let

$$\theta := \arctan(\gamma/2), \quad \dot{\theta} = \frac{2\dot{\gamma}}{\gamma^2 + 4}, \quad \theta \in [0, \pi/2). \tag{91}$$

The related kinematic quantities are listed as follows:

$$\mathbf{D} = \frac{\dot{\gamma}}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{W} = \frac{\dot{\gamma}}{2} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} 0 & \dot{\theta} & 0 \\ -\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{\Omega}_\mathrm{E} = \begin{bmatrix} 0 & \frac{\dot{\theta}}{2} & 0 \\ -\frac{\dot{\theta}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{L}_\mathrm{E} = \frac{\dot{\gamma}}{\gamma^2 + 4} \begin{bmatrix} \gamma & 3 & 0 \\ 1 & -\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{92}$$

$$\mathbf{\Omega}^{\log} = \frac{\dot{\gamma}}{2} \begin{bmatrix} 0 & f(\theta) & 0 \\ -f(\theta) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$f(\theta) = \frac{\sin\theta}{2\left(\ln\frac{1+\sin\theta}{\cos\theta}\right)} + \frac{\cos^2\theta}{2} \tag{93}$$

has been derived by Liu and Hong (1999). The material constants used in the calculations were $G = 50{,}000$ MPa, $\nu = 0.3$ and $\tau_\mathrm{y}^0 = 500$ MPa. The initial stresses were chosen to be located on the yield surface with $s^{11} = 300$ MPa, $s^{22} = 0$ MPa, $s^{23} = 0$ MPa, $s^{13} = 0$ MPa, and $s^{12} = 400$ MPa. Fig. 2 displays the stress response curves of the models of perfect elastoplasticity with the ten objective stress rates up to $\gamma = 1$. When γ is more larger, the results for different objective corotational stress rates deviate more pronounced. For the shear stress it can be seen that the ten curves are located in a narrow strip, having width about $0.0003\tau_\mathrm{y}^0$, between Szabó–Balla's and Sowerby–Chu's curves, indicating that the differences of objective stress rates have merely a minor influence on the shear stress. However, the non-spinning type objective stress rates result in very different axial stress responses as shown in Fig. 2(b). The rates of Szabó–Balla-1, Truesdell and Oldroyd all gave concave upwards curves with positive values, while the rates of Szabó–Balla-2 and Cotter–Rivlin gave negative slop curves with negative stress after some values of γ. It is only the corotational stress rates giving small positive axial stress and approaching to a narrow strip. This, as has been explained by Liu and Hong (2001), is due to the spinning values being far less than $1/\gamma_\mathrm{y}$, such that the matrix $\mathbf{A}$ in Eq. (52) is dominant by $1/\gamma_\mathrm{y}$ not by $\mathbf{A}_\mathrm{s}^\mathrm{s}$.

In order to compare the above three numerical schemes, let us consider a simple case with constant $\mathbf{D}$ and $\mathbf{W}$ as follows:

$$\mathbf{D} = \begin{bmatrix} 0.002 & 0.009 & 0.005 \\ 0.009 & -0.001 & 0.004 \\ 0.005 & 0.004 & -0.001 \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 0 & 0.001 & 0.002 \\ -0.001 & 0 & -0.005 \\ -0.002 & 0.005 & 0 \end{bmatrix}.$$

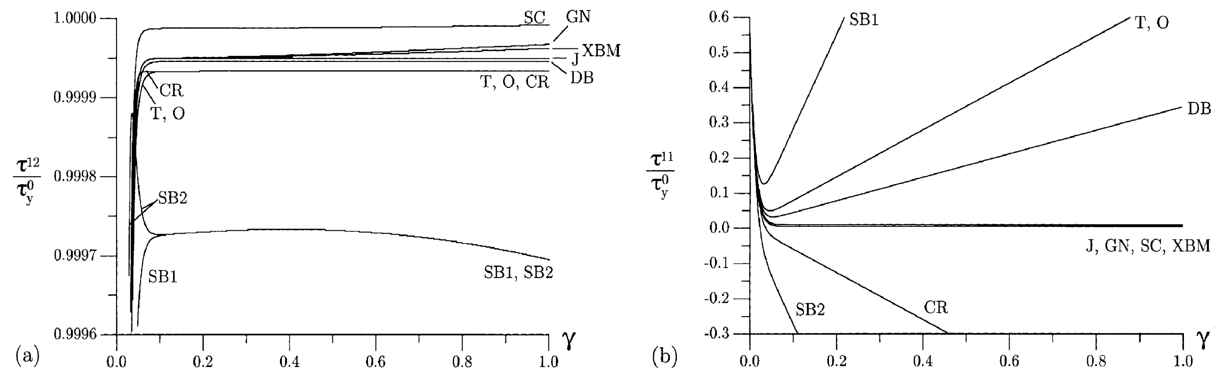

Fig. 2. The shear and axial stress responses for the simple shear problem are compared for: (1) Truesdell, (2) Oldroyd, (3) Cotter–Rivlin, (4) Jaumann, (5) Durban-Baruch, (6) Green–Naghdi, (7) Sowerby–Chu, (8) Szabó–Balla-1, (9) Szabó–Balla-2 and (10) Xiao–Bruhns–Meyers.

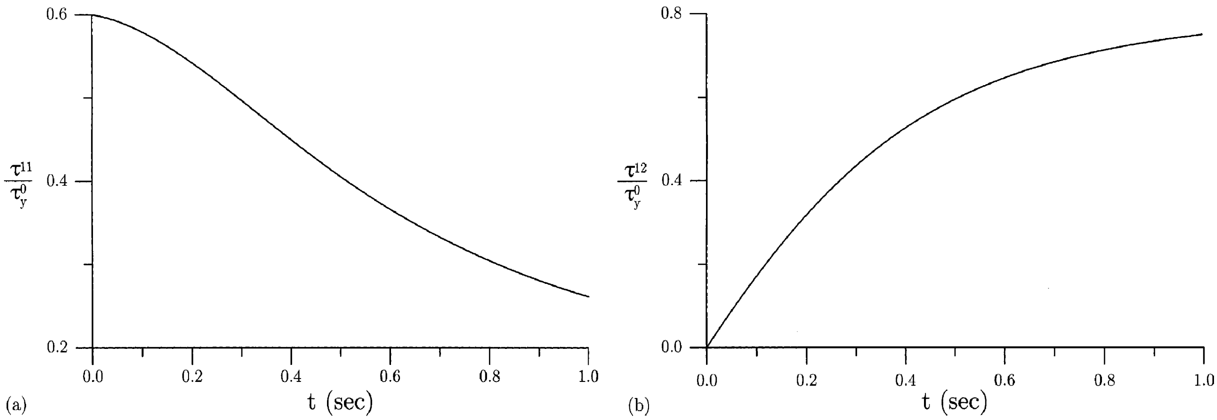

Fig. 3. Comparison of the numerical results calculated respectively by the schemes based on groups $SO_o(5,1)$, $SL(2,\mathbb{H})$ and $SU^*(4)$.

Here we employ the Jaumann stress rate in the model and calculate the stress responses with the initial conditions: $s^{11} = 300$ MPa, $s^{22} = 0$ MPa, $s^{23} = 400$ MPa, $s^{13} = 0$ MPa, and $s^{12} = 0$ MPa, which were chosen to be located on the yield surface. Fig. 3(a) and (b) shows that the above three numerical schemes gave almost the same results. Thus, it confirmed that the derivations of the group relations among $SU^*(4)$, $SL(2,\mathbb{H})$ and $SO_o(5,1)$ and their Lie algebras relations among $su^*(4)$, $sl(2,\mathbb{H})$ and $so(5,1)$ are correct.

The radial return method together with the first-order back Euler scheme is famous to approach the solution of plasticity model, which has good numerical performances and the well-established numerical properties. However, in order to match the consistent condition accurately the radial return method requires to solve a non-linear algebraic equation for the increment of $\Delta\lambda$ at each time stepping, and thus much computational time is spent for this work. For instance, the computational time of our schemes spent in the computation of the above numerical example is about 0.2 s, but the radial return method requires 2 s (with a prescribed error tolerance 10^{-3} of the consistent condition). Raising the order of accuracy increases the computational time correspondingly. Because our schemes satisfy the consistent condition automatically without any iteration, they can save about 90% or more CPU time than the conventional radial return method.

9. Conclusions

In this paper we have investigated the Lie symmetries inherent in the constitutive models of finite strain perfect elastoplasticity with different objective stress rates, which include two main types: spinning and non-spinning. Although the constitutive equations are highly non-linear in the deviatoric stress space of $\mathbf{s}$, as well as are coupled with the volumetric equation for the non-spinning objective stress rates, they can be converted to a Lie type system $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ in the $(5+1)$-dimensional augmented stress space of $\mathbf{X}$. In this space an internal spacetime structure of the Minkowskian type is brought out. The system matrix $\mathbf{A}$ for the plastic phase was proved to be an element of the real Lie algebra $so(5,1)$ of the proper orthochronous Lorentz group $SO_o(5,1)$, and the fundamental solution $\mathbf{G}$ of the system $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ with the plastic phase $\mathbf{A}$ was shown to be an element of the proper orthochronous Lorentz group.

Due to the nullity of $\mathbf{X}$ in the plastic phase we have further established a quaternionic two-component spinor representation. It is more economic than the Minkowski space representation due to its low dimensions. In the spinor space we obtained a governing equation $\dot{\boldsymbol{\alpha}} = \mathbf{Q}\boldsymbol{\alpha}$, and the underlying group was found to be $SL(2,\mathbb{H})$, which left acts on the spinor space $\boldsymbol{\alpha}$.

Moreover, the relations between the two groups $SL(2,\mathbb{H})$ and $SO_o(5,1)$, between their Lie algebras $sl(2,\mathbb{H})$ and $so(5,1)$, and between the systems (52) and (70) are explored in depth through the algebraic methods. This is of course due to the success of developing a new approach in Section 6. These exact relations (B.27)–(B.42), (C.33), (C.37) as well as (74) may be found their applications in several physical problems, not merely limited to the plasticity problem discussed here.

According to the symmetries studied in this paper, several numerical schemes which preserve the group properties for every time increment were developed. This group preserving scheme may be specifically called an exactly consistent scheme, since it is capable, among other benefits derivable from the group properties, of updating the stress point automatically located on the yield surface at the end of each time increment in the plastic phase without any iterative calculations, that is, the consistency condition is fulfilled automatically and exactly. In this regard, the conventional numerical schemes typically do not share the group properties so that perform less accurate than the consistency scheme. Since the new scheme is easy to implement numerically and has high computational efficiency and high accuracy, it is recommended to be used in engineering applications which may require intensive calculations.

Simple shearings are calculated to compare 10 corotational and non-corotational stress rates suggested in the literature. The results show that, for simple shear deformation, the corotational stress rates supply reasonable responses for both shear and normal stress components, whereas the non-corotational stress rates provide unrealistic normal stress responses which are not small and not tending to zero, but grow unlimited with increasing shear strain. In addition to these drawbacks, the use of non-corotational stress rates in the constitutive equations increases their non-linearity degree one than the use of corotational stress rates, and also makes the coupling of deviatoric and volumetric constitutive equations. On the other hand, from a computational view the two advantages of uncoupling and linearity in augmented stress space strongly support the use of corotational stress rates in the finite strain constitutive equations, and they also match the Prager yielding-stationarity criterion.

Acknowledgements

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Appendix A. Quaternion algebra

The quaternions usually defined as a four-dimensional real vector space such that we can define a product $(\mathbf{x},\mathbf{y}) \to \mathbf{xy}$ satisfying the following associative and distributive laws for all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{H}$ and all $a \in \mathbb{R}$ (see, e.g., Okubo, 1995):

$$(\mathbf{xy})\mathbf{z} = \mathbf{x}(\mathbf{yz}), \tag{A.1}$$

$$\mathbf{x}(\mathbf{y}+\mathbf{z}) = \mathbf{xy}+\mathbf{xz}, \tag{A.2}$$

$$(\mathbf{x}+\mathbf{y})\mathbf{z} = \mathbf{xz}+\mathbf{yz}, \tag{A.3}$$

$$a(\mathbf{xy}) = (a\mathbf{x})\mathbf{y} = \mathbf{x}(a\mathbf{y}). \tag{A.4}$$

There exists a distinguished basis elements $\{1,\mathbf{i}_2,\mathbf{i}_3,\mathbf{i}_4\}$ with the following commutation relations:

$$\begin{aligned}&\mathbf{i}_2^2 = \mathbf{i}_3^2 = \mathbf{i}_4^2 = -1,\\&\mathbf{i}_2\mathbf{i}_3 = -\mathbf{i}_3\mathbf{i}_2 = \mathbf{i}_4, \mathbf{i}_3\mathbf{i}_4 = -\mathbf{i}_4\mathbf{i}_3 = \mathbf{i}_2, \mathbf{i}_4\mathbf{i}_2 = -\mathbf{i}_2\mathbf{i}_4 = \mathbf{i}_3.\end{aligned} \tag{A.5}$$

Thus, if $\mathbf{x} = x^1 + x^2\mathbf{i}_2 + x^3\mathbf{i}_3 + x^4\mathbf{i}_4$ and $\mathbf{y} = y^1 + y^2\mathbf{i}_2 + y^3\mathbf{i}_3 + y^4\mathbf{i}_4$ are any two quaternions, their product is defined by

$$\begin{aligned}\mathbf{xy} = {}&[x^1y^1 - x^2y^2 - x^3y^3 - x^4y^4] + [x^1y^2 + y^1x^2 + x^3y^4 - x^4y^3]\mathbf{i}_2 + [x^1y^3 + y^1x^3 + x^4y^2 - x^2y^4]\mathbf{i}_3\\&+ [x^1y^4 + y^1x^4 + x^2y^3 - x^3y^2]\mathbf{i}_4.\end{aligned} \tag{A.6}$$

The conjugate of $\mathbf{x}$ is denoted by $\bar{\mathbf{x}} := x^1 - x^2\mathbf{i}_2 - x^3\mathbf{i}_3 - x^4\mathbf{i}_4$, such that the product of $\mathbf{x}$ and $\bar{\mathbf{x}}$ gives the usual squared norm of $\mathbf{x}$ in $\mathbb{E}^4$

$$\|\mathbf{x}\|^2 = \mathbf{x}\bar{\mathbf{x}} = \bar{\mathbf{x}}\mathbf{x} = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2. \tag{A.7}$$

For non-zero quaternion the inverse $\mathbf{x}^{-1}$ is thus given by

$$\mathbf{x}^{-1} = \frac{\bar{\mathbf{x}}}{\|\mathbf{x}\|^2}. \tag{A.8}$$

If we let $\mathbf{x} = x^1 + \mathbf{x}^s$, $\mathbf{y} = y^1 + \mathbf{y}^s$, it is shown by Liu (2002) that the product rule (A.6) can be represented by

$$\mathbf{xy} = x^1y^1 - \mathbf{x}^s\cdot\mathbf{y}^s + x^1\mathbf{y}^s + y^1\mathbf{x}^s + \mathbf{x}^s\times\mathbf{y}^s = x^1y^1 - \mathbf{x}^s\cdot\mathbf{y}^s + x^1\mathbf{y}^s + y^1\mathbf{x}^s + \tilde{\mathbf{x}}^s\mathbf{y}^s, \tag{A.9}$$

where the cross-product of $\mathbf{x}^s\times\mathbf{y}^s$ and the inner product of $\mathbf{x}^s\cdot\mathbf{y}^s$ are defined in the three-dimensional Euclidean space, and

$$\sim: \mathbf{x}^s \mapsto \tilde{\mathbf{x}}^s := \begin{bmatrix} 0 & -x^4 & x^3\\ x^4 & 0 & -x^2\\ -x^3 & x^2 & 0\end{bmatrix} \tag{A.10}$$

is the tilde mapping, which maps each axial vector $\mathbf{x}^s := (x^2,x^3,x^4)^{\mathrm{T}}$ to a skew-symmetric matrix $\tilde{\mathbf{x}}^s$.

We also need to define the scalar product of two quaternions. This can be achieved through the inner product of the quaternion bases

$$\mathbf{i}_j\cdot\mathbf{i}_k = \delta_{jk}, \quad j,k = 1,2,3,4, \tag{A.11}$$

where $\mathbf{i}_1$ denotes the unit element 1 of the quaternion, and δ_{jk} is the Kronecker delta function. So that we have

$$\mathbf{x}\cdot\mathbf{y} = x^1y^1 + \mathbf{x}^s\cdot\mathbf{y}^s = x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4,$$
$$\mathbf{x}\cdot\bar{\mathbf{y}} = x^1y^1 - \mathbf{x}^s\cdot\mathbf{y}^s = x^1y^1 - x^2y^2 - x^3y^3 - x^4y^4 \quad \text{(A.12)}$$

and accordingly, Eq. (A.9) can be written as

$$\mathbf{x}\mathbf{y} = \mathbf{x}\cdot\bar{\mathbf{y}} + x^1\mathbf{y}^s + y^1\mathbf{x}^s + \mathbf{x}^s\times\mathbf{y}^s. \quad \text{(A.13)}$$

For the later use we further derive the following formula for the product of three quaternions:

$$\mathbf{x}\mathbf{y}\mathbf{z} = z^1\mathbf{x}\cdot\bar{\mathbf{y}} - x^1\mathbf{y}^s\cdot\mathbf{z}^s - y^1\mathbf{x}^s\cdot\mathbf{z}^s - \mathbf{x}^s\times\mathbf{y}^s\cdot\mathbf{z}^s + \mathbf{x}\cdot\bar{\mathbf{y}}\mathbf{z}^s + \mathbf{x}\cdot\mathbf{z}\mathbf{y}^s + \mathbf{z}\cdot\bar{\mathbf{y}}\mathbf{x}^s + z^1\mathbf{x}^s\times\mathbf{y}^s$$
$$+ x^1\mathbf{y}^s\times\mathbf{z}^s + y^1\mathbf{x}^s\times\mathbf{z}^s. \quad \text{(A.14)}$$

Appendix B. Spinor map from $SL(2,\mathbb{H})$ to $SO_o(5,1)$

Dirac has proposed a quaternionic representation of the Lorentz transformation in $\mathbb{M}^{3+1}$ by expressing the quaternion **w** as the ratio of the other two quaternions **u** and **v** (see, e.g., Dirac, 1945)

$$\mathbf{w} = \mathbf{u}\mathbf{v}^{-1}. \quad \text{(B.1)}$$

Then he considered three quantities

$$\mathbf{x} = \mathbf{u}\bar{\mathbf{v}}, \quad x^+ = \mathbf{v}\bar{\mathbf{v}}, \quad x^- = \mathbf{u}\bar{\mathbf{u}}, \quad \text{(B.2)}$$

and defined

$$\mathbf{x} = x^1 + x^2\mathbf{i}_2 + x^3\mathbf{i}_3 + x^4\mathbf{i}_4, \quad x^+ = x^0 + x^5, \quad x^- = x^0 - x^5, \quad \text{(B.3)}$$

where $x^1,\dots,x^5,x^0$ are real numbers. If **u** and **v** are replaced by $\mathbf{u}\boldsymbol{\lambda}$ and $\mathbf{v}\boldsymbol{\lambda}$, the six x's all get multiplied by $\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}$ and their ratios are unchanged. Thus the ratios of x's are determined by **w**. From Eq. (B.2) it follows that

$$\mathbf{x}\bar{\mathbf{x}} = \mathbf{u}\bar{\mathbf{v}}\mathbf{v}\bar{\mathbf{u}} = \mathbf{u}x^+\bar{\mathbf{u}} = x^+x^-, \quad \text{(B.4)}$$

which by means of Eqs. (A.7) and (B.3) leads to

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = (x^0)^2 - (x^5)^2. \quad \text{(B.5)}$$

It represents a null cone in the space $\mathbb{M}^{5+1}$.

Under the following linear transformations for **u** and **v**,

$$\hat{\mathbf{u}} = \mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}, \quad \hat{\mathbf{v}} = \mathbf{c}\mathbf{u} + \mathbf{d}\mathbf{v}, \quad \text{(B.6)}$$

where **a**, **b**, **c**, and **d** are arbitrary quaternions, **x**, x^+, and x^- are transformed as follows:

$$\hat{\mathbf{x}} = \mathbf{a}x^-\bar{\mathbf{c}} + \mathbf{b}x^+\bar{\mathbf{d}} + \mathbf{a}\mathbf{x}\bar{\mathbf{d}} + \mathbf{b}\bar{\mathbf{x}}\bar{\mathbf{c}}, \quad \text{(B.7)}$$

$$\hat{x}^+ = \mathbf{c}x^-\bar{\mathbf{c}} + \mathbf{d}x^+\bar{\mathbf{d}} + \mathbf{c}\mathbf{x}\bar{\mathbf{d}} + \mathbf{d}\bar{\mathbf{x}}\bar{\mathbf{c}}, \quad \text{(B.8)}$$

$$\hat{x}^- = \mathbf{a}x^-\bar{\mathbf{a}} + \mathbf{b}x^+\bar{\mathbf{b}} + \mathbf{a}\mathbf{x}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{x}}\bar{\mathbf{a}}. \quad \text{(B.9)}$$

This, as has been demonstrated by Dirac, will result in the new six $\hat{x}$'s being linear functions of the old x's, and Eq. (B.5) still holds for the new $\hat{x}$'s.

Eqs. (B.7)–(B.9) together with the conjugate of Eq. (B.7) can be represented by

$$\begin{bmatrix} \hat{x}^+ & \hat{\bar{\mathbf{x}}} \\ \hat{\mathbf{x}} & \hat{x}^- \end{bmatrix} = \begin{bmatrix} \mathbf{d} & \mathbf{c} \\ \mathbf{b} & \mathbf{a} \end{bmatrix} \begin{bmatrix} x^+ & \bar{\mathbf{x}} \\ \mathbf{x} & x^- \end{bmatrix} \begin{bmatrix} \bar{\mathbf{d}} & \bar{\mathbf{b}} \\ \bar{\mathbf{c}} & \bar{\mathbf{a}} \end{bmatrix}. \quad \text{(B.10)}$$

By letting

$$\mathbf{U} = \begin{bmatrix} \mathbf{d} & \mathbf{c} \\ \mathbf{b} & \mathbf{a} \end{bmatrix}, \tag{B.11}$$

Eq. (B.10) is really of the form (67).

Although the product $\mathbf{xy}$ is non-commutative, by Eq. (A.9) we can derive the following relation:

$$\mathbf{xy} = \mathbf{yx} - 2\langle\mathbf{yx}\rangle = \mathbf{yx} - 2\tilde{\mathbf{y}}^s\mathbf{x}^s, \tag{B.12}$$

where $\langle\mathbf{yx}\rangle$ denotes the skew-symmetric (cross product) part of $\mathbf{yx}$. With this formula we can rearrange Eqs. (B.7)–(B.9) and the conjugate of Eq. (B.7) to the following forms:

$$\hat{\mathbf{x}} = \mathbf{a}\bar{\mathbf{c}}x^- + \mathbf{b}\bar{\mathbf{d}}x^+ + \mathbf{a}\bar{\mathbf{d}}\mathbf{x} - 2\mathbf{a}\langle\bar{\mathbf{d}}\mathbf{x}\rangle + \mathbf{b}\bar{\mathbf{c}}\bar{\mathbf{x}} - 2\mathbf{b}\langle\bar{\mathbf{c}}\bar{\mathbf{x}}\rangle, \tag{B.13}$$

$$\hat{\bar{\mathbf{x}}} = \mathbf{c}\bar{\mathbf{a}}x^- + \mathbf{d}\bar{\mathbf{b}}x^+ + \mathbf{d}\bar{\mathbf{a}}\bar{\mathbf{x}} - 2\mathbf{d}\langle\bar{\mathbf{a}}\bar{\mathbf{x}}\rangle + \mathbf{c}\bar{\mathbf{b}}\mathbf{x} - 2\mathbf{c}\langle\bar{\mathbf{b}}\mathbf{x}\rangle, \tag{B.14}$$

$$\hat{x}^+ = \mathbf{c}\bar{\mathbf{c}}x^- + \mathbf{d}\bar{\mathbf{d}}x^+ + \mathbf{c}\bar{\mathbf{d}}\mathbf{x} - 2\mathbf{c}\langle\bar{\mathbf{d}}\mathbf{x}\rangle + \mathbf{d}\bar{\mathbf{c}}\bar{\mathbf{x}} - 2\mathbf{d}\langle\bar{\mathbf{c}}\bar{\mathbf{x}}\rangle, \tag{B.15}$$

$$\hat{x}^- = \mathbf{a}\bar{\mathbf{a}}x^- + \mathbf{b}\bar{\mathbf{b}}x^+ + \mathbf{a}\bar{\mathbf{b}}\mathbf{x} - 2\mathbf{a}\langle\bar{\mathbf{b}}\mathbf{x}\rangle + \mathbf{b}\bar{\mathbf{a}}\bar{\mathbf{x}} - 2\mathbf{b}\langle\bar{\mathbf{a}}\bar{\mathbf{x}}\rangle. \tag{B.16}$$

The above four equations can be combined together to a matrix representation

$$\begin{bmatrix} \hat{x}^+ \\ \hat{\bar{\mathbf{x}}} \\ \hat{\mathbf{x}} \\ \hat{x}^- \end{bmatrix} = \mathbf{J} \begin{bmatrix} x^+ \\ \bar{\mathbf{x}} \\ \mathbf{x} \\ x^- \end{bmatrix}, \tag{B.17}$$

where

$$\mathbf{J} := \begin{bmatrix} \mathbf{d}\bar{\mathbf{d}} & \mathbf{d}\bar{\mathbf{c}} - 2\mathbf{d}\langle\bar{\mathbf{c}} & \mathbf{c}\bar{\mathbf{d}} - 2\mathbf{c}\langle\bar{\mathbf{d}} & \mathbf{c}\bar{\mathbf{c}} \\ \mathbf{d}\bar{\mathbf{b}} & \mathbf{d}\bar{\mathbf{a}} - 2\mathbf{d}\langle\bar{\mathbf{a}} & \mathbf{c}\bar{\mathbf{b}} - 2\mathbf{c}\langle\bar{\mathbf{b}} & \mathbf{c}\bar{\mathbf{a}} \\ \mathbf{b}\bar{\mathbf{d}} & \mathbf{b}\bar{\mathbf{c}} - 2\mathbf{b}\langle\bar{\mathbf{c}} & \mathbf{a}\bar{\mathbf{d}} - 2\mathbf{a}\langle\bar{\mathbf{d}} & \mathbf{a}\bar{\mathbf{c}} \\ \mathbf{b}\bar{\mathbf{b}} & \mathbf{b}\bar{\mathbf{a}} - 2\mathbf{b}\langle\bar{\mathbf{a}} & \mathbf{a}\bar{\mathbf{b}} - 2\mathbf{a}\langle\bar{\mathbf{b}} & \mathbf{a}\bar{\mathbf{a}} \end{bmatrix}, \tag{B.18}$$

and $\langle\cdot$ denotes the operator of skew-symmetrization; for example, the operator $\langle\mathbf{y}$ acting on $\mathbf{x}$ is read as $\langle\mathbf{yx}\rangle = \tilde{\mathbf{y}}^s\mathbf{x}^s$.

For the later purpose we introduce another representation of the quaternions $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x} = x^1 + \mathrm{i}\mathbf{x}^s$ and $\mathbf{y} = y^1 + \mathrm{i}\mathbf{y}^s$, such that their product is expressed by

$$\mathbf{xy} = (x^1 + \mathrm{i}\mathbf{x}^s)(y^1 + \mathrm{i}\mathbf{y}^s) := x^1y^1 - \mathbf{x}^s \cdot \mathbf{y}^s + \mathrm{i}(x^1\mathbf{y}^s + y^1\mathbf{x}^s + \mathbf{x}^s \times \mathbf{y}^s). \tag{B.19}$$

Here 'i' plays not only the role of an imaginary number with $\mathrm{i}^2 = -1$, but also a symbol used to stress that the quantity been prefixed by 'i' is the spatial part; for example, $\mathbf{x}^s$ is the spatial part of $\mathbf{x}$; conversely, x^1 is the scalar part of $\mathbf{x}$.

Now, the $\mathbf{H}$ in Eq. (68), with its $x^0 + x^5$ replaced by x^+ and $x^0 - x^5$ by x^- as that defined in Eq. (B.3), can be re-expressed as

$$\begin{bmatrix} x^+ \\ \bar{\mathbf{x}} \\ \mathbf{x} \\ x^- \end{bmatrix} = \mathbf{C} \begin{bmatrix} x^1 \\ \mathbf{x}^s \\ x^5 \\ x^0 \end{bmatrix}, \tag{B.20}$$

where

$$\mathbf{C} := \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & -\mathrm{i} & 0 & 0 \\ 1 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{B.21}$$

Conversely, we have

$$\mathbf{X} := \begin{bmatrix} x^1 \\ \mathbf{x}^{\mathrm{s}} \\ x^5 \\ x^0 \end{bmatrix} = \mathbf{C}^{-1} \begin{bmatrix} x^+ \\ \bar{\mathbf{x}} \\ \mathbf{x} \\ x^- \end{bmatrix}, \tag{B.22}$$

where

$$\mathbf{C}^{-1} := \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & \mathrm{i} & -\mathrm{i} & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{B.23}$$

Left multiplying both sides of Eq. (B.17) by $\mathbf{C}^{-1}$ and noting (B.20), yields

$$\widehat{\mathbf{X}} = \mathbf{C}^{-1}\mathbf{J}\mathbf{C}\mathbf{X}, \tag{B.24}$$

which being compared with the proper orthochronous Lorentz transformation $\widehat{\mathbf{X}} = \mathbf{G}\mathbf{X}$ gives

$$\mathbf{G} = \mathbf{C}^{-1}\mathbf{J}\mathbf{C}. \tag{B.25}$$

Now, letting

$$\mathbf{G} := \begin{bmatrix} G^1_1 & \mathbf{G}^1_{\mathrm{s}} & G^1_5 & G^1_0 \\ \mathbf{G}^{\mathrm{s}}_1 & \mathbf{G}^{\mathrm{s}}_{\mathrm{s}} & \mathbf{G}^{\mathrm{s}}_5 & \mathbf{G}^{\mathrm{s}}_0 \\ G^5_1 & \mathbf{G}^5_{\mathrm{s}} & G^5_5 & G^5_0 \\ G^0_1 & \mathbf{G}^0_{\mathrm{s}} & G^0_5 & G^0_0 \end{bmatrix}, \tag{B.26}$$

and then substituting Eq. (B.23) for $\mathbf{C}^{-1}$, Eq. (B.18) for $\mathbf{J}$ and Eq. (B.21) for $\mathbf{C}$ into Eq. (B.25), we obtain each term in $\mathbf{G}$ as follows:

$$G^1_1 = \frac{1}{2}(\mathbf{a}\bar{\mathbf{d}} + \mathbf{d}\bar{\mathbf{a}} + \mathbf{b}\bar{\mathbf{c}} + \mathbf{c}\bar{\mathbf{b}}) - \mathbf{d}\langle\bar{\mathbf{a}}1\rangle - \mathbf{b}\langle\bar{\mathbf{c}}1\rangle - \mathbf{c}\langle\bar{\mathbf{b}}1\rangle - \mathbf{a}\langle\bar{\mathbf{d}}1\rangle = \mathbf{a}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{c}, \tag{B.27}$$

$$\begin{aligned} \mathbf{G}^1_{\mathrm{s}} &= \frac{\mathrm{i}}{2}(\mathbf{a}\bar{\mathbf{d}} - \mathbf{d}\bar{\mathbf{a}} + \mathbf{c}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{c}}) + \mathbf{d}\langle\bar{\mathbf{a}}\mathrm{i}\rangle + \mathbf{b}\langle\bar{\mathbf{c}}\mathrm{i}\rangle - \mathbf{c}\langle\bar{\mathbf{b}}\mathrm{i}\rangle - \mathbf{a}\langle\bar{\mathbf{d}}\mathrm{i}\rangle \\ &= (a_1\mathbf{d}_{\mathrm{s}} - d_1\mathbf{a}_{\mathrm{s}} + c_1\mathbf{b}_{\mathrm{s}} - b_1\mathbf{c}_{\mathrm{s}} + \mathbf{d}_{\mathrm{s}}\times\mathbf{a}_{\mathrm{s}} + \mathbf{b}_{\mathrm{s}}\times\mathbf{c}_{\mathrm{s}})^{\mathrm{T}}, \end{aligned} \tag{B.28}$$

$$G^1_5 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{d}} - \mathbf{c}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{c}}), \tag{B.29}$$

$$G^1_0 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{d}} + \mathbf{c}\bar{\mathbf{a}} + \mathbf{a}\bar{\mathbf{c}}), \tag{B.30}$$

$$\begin{aligned} \mathbf{G}^{\mathrm{s}}_1 &= \frac{\mathrm{i}}{2}(\mathbf{d}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{d}} + \mathbf{c}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{c}}) + \mathrm{i}(\mathbf{a}\langle\bar{\mathbf{d}}1\rangle + \mathbf{b}\langle\bar{\mathbf{c}}1\rangle - \mathbf{c}\langle\bar{\mathbf{b}}1\rangle - \mathbf{d}\langle\bar{\mathbf{a}}1\rangle) \\ &= c_1\mathbf{b}_{\mathrm{s}} + d_1\mathbf{a}_{\mathrm{s}} - a_1\mathbf{d}_{\mathrm{s}} - b_1\mathbf{c}_{\mathrm{s}} - \mathbf{a}_{\mathrm{s}}\times\mathbf{d}_{\mathrm{s}} - \mathbf{b}_{\mathrm{s}}\times\mathbf{c}_{\mathrm{s}}, \end{aligned} \tag{B.31}$$

$$\mathbf{G}_s^s = \frac{1}{2}(\mathbf{a}\bar{\mathbf{d}} + \mathbf{d}\bar{\mathbf{a}} - \mathbf{b}\bar{\mathbf{c}} - \mathbf{c}\bar{\mathbf{b}}) + \mathrm{i}(\mathbf{a}\langle\bar{\mathbf{d}}\mathrm{i}\rangle + \mathbf{d}\langle\bar{\mathbf{a}}\mathrm{i}\rangle - \mathbf{b}\langle\bar{\mathbf{c}}\mathrm{i}\rangle - \mathbf{c}\langle\bar{\mathbf{b}}\mathrm{i}\rangle)$$
$$= (\mathbf{a}\cdot\bar{\mathbf{d}} - \mathbf{b}\cdot\bar{\mathbf{c}})\mathbf{I}_3 + 2[\mathbf{a}_s\otimes\mathbf{d}_s] - 2[\mathbf{b}_s\otimes\mathbf{c}_s] + d_1\tilde{\mathbf{a}}_s + a_1\tilde{\mathbf{d}}_s - c_1\tilde{\mathbf{b}}_s - b_1\tilde{\mathbf{c}}_s, \tag{B.32}$$

$$\mathbf{G}_5^s = \frac{\mathrm{i}}{2}(\mathbf{d}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{d}} + \mathbf{a}\bar{\mathbf{c}} - \mathbf{c}\bar{\mathbf{a}}) = d_1\mathbf{b}_s - b_1\mathbf{d}_s - c_1\mathbf{a}_s + a_1\mathbf{c}_s + \mathbf{d}_s\times\mathbf{b}_s - \mathbf{c}_s\times\mathbf{a}_s, \tag{B.33}$$

$$\mathbf{G}_0^s = \frac{\mathrm{i}}{2}(\mathbf{d}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{d}} + \mathbf{c}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{c}}) = d_1\mathbf{b}_s - b_1\mathbf{d}_s + c_1\mathbf{a}_s - a_1\mathbf{c}_s + \mathbf{d}_s\times\mathbf{b}_s + \mathbf{c}_s\times\mathbf{a}_s, \tag{B.34}$$

$$G_1^5 = \frac{1}{2}(\mathbf{c}\bar{\mathbf{d}} + \mathbf{d}\bar{\mathbf{c}} - \mathbf{a}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{a}}) - \mathbf{d}\langle\bar{\mathbf{c}}1\rangle + \mathbf{b}\langle\bar{\mathbf{a}}1\rangle - \mathbf{c}\langle\bar{\mathbf{d}}1\rangle + \mathbf{a}\langle\bar{\mathbf{b}}1\rangle = \mathbf{c}\cdot\mathbf{d} - \mathbf{a}\cdot\mathbf{b}, \tag{B.35}$$

$$\mathbf{G}_s^5 = \frac{\mathrm{i}}{2}(\mathbf{b}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{b}} + \mathbf{c}\bar{\mathbf{d}} - \mathbf{d}\bar{\mathbf{c}}) + \mathbf{d}\langle\bar{\mathbf{c}}\mathrm{i}\rangle - \mathbf{b}\langle\bar{\mathbf{a}}\mathrm{i}\rangle - \mathbf{c}\langle\bar{\mathbf{d}}\mathrm{i}\rangle + \mathbf{a}\langle\bar{\mathbf{b}}\mathrm{i}\rangle$$
$$= (c_1\mathbf{d}_s - d_1\mathbf{c}_s - a_1\mathbf{b}_s + b_1\mathbf{a}_s + \mathbf{d}_s\times\mathbf{c}_s - \mathbf{b}_s\times\mathbf{a}_s)^{\mathrm{T}}, \tag{B.36}$$

$$G_5^5 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{d}} - \mathbf{b}\bar{\mathbf{b}} - \mathbf{c}\bar{\mathbf{c}} + \mathbf{a}\bar{\mathbf{a}}), \tag{B.37}$$

$$G_0^5 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{d}} - \mathbf{b}\bar{\mathbf{b}} + \mathbf{c}\bar{\mathbf{c}} - \mathbf{a}\bar{\mathbf{a}}), \tag{B.38}$$

$$G_1^0 = \frac{1}{2}(\mathbf{a}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{a}} + \mathbf{c}\bar{\mathbf{d}} + \mathbf{d}\bar{\mathbf{c}}) - \mathbf{d}\langle\bar{\mathbf{c}}1\rangle - \mathbf{b}\langle\bar{\mathbf{a}}1\rangle - \mathbf{c}\langle\bar{\mathbf{d}}1\rangle - \mathbf{a}\langle\bar{\mathbf{b}}1\rangle = \mathbf{c}\cdot\mathbf{d} + \mathbf{a}\cdot\mathbf{b}, \tag{B.39}$$

$$\mathbf{G}_s^0 = \frac{\mathrm{i}}{2}(\mathbf{a}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{a}} + \mathbf{c}\bar{\mathbf{d}} - \mathbf{d}\bar{\mathbf{c}}) + \mathbf{d}\langle\bar{\mathbf{c}}\mathrm{i}\rangle + \mathbf{b}\langle\bar{\mathbf{a}}\mathrm{i}\rangle - \mathbf{c}\langle\bar{\mathbf{d}}\mathrm{i}\rangle - \mathbf{a}\langle\bar{\mathbf{b}}\mathrm{i}\rangle$$
$$= (c_1\mathbf{d}_s - d_1\mathbf{c}_s + a_1\mathbf{b}_s - b_1\mathbf{a}_s + \mathbf{d}_s\times\mathbf{c}_s + \mathbf{b}_s\times\mathbf{a}_s)^{\mathrm{T}}, \tag{B.40}$$

$$G_5^0 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{d}} + \mathbf{b}\bar{\mathbf{b}} - \mathbf{c}\bar{\mathbf{c}} - \mathbf{a}\bar{\mathbf{a}}), \tag{B.41}$$

$$G_0^0 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{d}} + \mathbf{b}\bar{\mathbf{b}} + \mathbf{c}\bar{\mathbf{c}} + \mathbf{a}\bar{\mathbf{a}}). \tag{B.42}$$

In above $\otimes$ between two three-dimensional vectors denotes their tensor product, and as before $[\mathbf{a}_s\otimes\mathbf{d}_s]$ and $[\mathbf{b}_s\otimes\mathbf{c}_s]$ denote, respectively, the symmetric parts of the tensors $\mathbf{a}_s\otimes\mathbf{d}_s$ and $\mathbf{b}_s\otimes\mathbf{c}_s$.

Only G_5^1, G_0^1, G_5^5, G_0^5, G_5^0 and G_0^0 are calculable directly from the quaternions **a**, **b**, **c** and **d**. The other terms above should be supplemented with their second equalities for directly calculable from the quaternions **a**, **b**, **c** and **d**. We first derive the second equality in $\mathbf{G}_s^s$. Using the following formulae:

$$\mathbf{x}^s\times(\mathbf{y}^s\times\mathbf{z}^s) = (\mathbf{x}^s\cdot\mathbf{z}^s)\mathbf{y}^s - (\mathbf{x}^s\cdot\mathbf{y}^s)\mathbf{z}^s, \tag{B.43}$$

$$\mathbf{x}^s\cdot(\mathbf{y}^s\times\mathbf{z}^s) = \mathbf{y}^s\cdot(\mathbf{z}^s\times\mathbf{x}^s) = \mathbf{z}^s\cdot(\mathbf{x}^s\times\mathbf{y}^s) \tag{B.44}$$

for three-dimensional vectors $\mathbf{x}^s$, $\mathbf{y}^s$ and $\mathbf{z}^s$, we can prove that

$$\begin{aligned}\mathrm{i}\mathbf{a}\langle\bar{\mathbf{d}}\mathrm{i}\mathbf{x}\rangle + \mathrm{i}\mathbf{d}\langle\bar{\mathbf{a}}\,\mathrm{i}\mathbf{x}\rangle &= -\mathbf{a}_s\cdot(\mathbf{d}_s\times\mathbf{x}^s) + a_1\tilde{\mathbf{d}}_s\mathbf{x}^s + \mathbf{a}_s\cdot\mathbf{x}^s\mathbf{d}_s - \mathbf{a}_s\cdot\mathbf{d}_s\mathbf{x}^s \\ &\quad - \mathbf{d}_s\cdot(\mathbf{a}_s\times\mathbf{x}^s) + d_1\tilde{\mathbf{a}}_s\mathbf{x}^s + \mathbf{d}_s\cdot\mathbf{x}^s\mathbf{a}_s - \mathbf{d}_s\cdot\mathbf{a}_s\mathbf{x}^s \\ &= a_1\tilde{\mathbf{d}}_s\mathbf{x}^s + d_1\tilde{\mathbf{a}}_s\mathbf{x}^s + 2[\mathbf{a}_s\otimes\mathbf{d}_s]\mathbf{x}^s - 2\mathbf{a}_s\cdot\mathbf{d}_s\mathbf{x}^s.\end{aligned} \tag{B.45}$$

It thus follows that

$$\mathrm{i}\mathbf{a}\langle\bar{\mathbf{d}}\mathrm{i}\rangle + \mathrm{i}\mathbf{d}\langle\bar{\mathbf{a}}\mathrm{i}\rangle = a_1\tilde{\mathbf{d}}_s + d_1\tilde{\mathbf{a}}_s + 2[\mathbf{a}_s\otimes\mathbf{d}_s] - 2\mathbf{a}_s\cdot\mathbf{d}_s\mathbf{I}_3 \tag{B.46}$$

and similarly,

$$-\mathrm{i}\mathbf{b}\langle\bar{\mathbf{c}}\mathrm{i}\rangle - \mathrm{i}\mathbf{c}\langle\bar{\mathbf{b}}\mathrm{i}\rangle = -b_1\tilde{\mathbf{c}}_s - c_1\tilde{\mathbf{b}}_s - 2[\mathbf{b}_s\otimes\mathbf{c}_s] + 2\mathbf{b}_s\cdot\mathbf{c}_s\mathbf{I}_3. \tag{B.47}$$

While the other terms in $\mathbf{G}_s^s$ can be computed as follows:

$$-\mathrm{i}\mathbf{a}\bar{\mathbf{d}}\mathrm{i} - \mathrm{i}\mathbf{d}\bar{\mathbf{a}}\mathrm{i} + \mathrm{i}\mathbf{b}\bar{\mathbf{c}}\mathrm{i} + \mathrm{i}\mathbf{c}\bar{\mathbf{b}}\mathrm{i} = (2a_1d_1 + 2\mathbf{a}_s\cdot\mathbf{d}_s - 2b_1c_1 - 2\mathbf{b}_s\cdot\mathbf{c}_s)\mathbf{I}_3. \tag{B.48}$$

Substituting Eqs. (B.46)–(B.48) into the first equality on the right-hand side of $\mathbf{G}_s^s$, we obtain the second equality in Eq. (B.32) for $\mathbf{G}_s^s$.

There are

$$\langle\bar{\mathbf{a}}1\rangle = 0, \quad \langle\bar{\mathbf{b}}1\rangle = 0, \quad \langle\bar{\mathbf{c}}1\rangle = 0, \quad \langle\bar{\mathbf{d}}1\rangle = 0,$$

because 1 is a scalar; see the sentence follows Eq. (B.18). Thus, the second equalities of G_1^1, G_1^5 and G_1^0 in Eqs. (B.27), (B.35) and (B.39) are proved.

By straightforward calculations we can prove that

$$\mathrm{i}(\mathbf{a}\bar{\mathbf{d}} - \mathbf{d}\bar{\mathbf{a}}) = 2(a_1\mathbf{d}_s - d_1\mathbf{a}_s + \mathbf{a}_s\times\mathbf{d}_s), \tag{B.49}$$

$$\mathrm{i}(\mathbf{c}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{c}}) = 2(c_1\mathbf{b}_s - b_1\mathbf{c}_s + \mathbf{c}_s\times\mathbf{b}_s). \tag{B.50}$$

Therefore the following equations hold:

$$\mathrm{i}(\mathbf{d}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{d}} + \mathbf{c}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{c}}) = 2(d_1\mathbf{a}_s - a_1\mathbf{d}_s - \mathbf{a}_s\times\mathbf{d}_s) + 2(c_1\mathbf{b}_s - b_1\mathbf{c}_s + \mathbf{c}_s\times\mathbf{b}_s),$$

$$\mathrm{i}(\mathbf{d}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{d}} + \mathbf{a}\bar{\mathbf{c}} - \mathbf{c}\bar{\mathbf{a}}) = 2(d_1\mathbf{b}_s - b_1\mathbf{d}_s - \mathbf{b}_s\times\mathbf{d}_s) + 2(a_1\mathbf{c}_s - c_1\mathbf{a}_s - \mathbf{c}_s\times\mathbf{a}_s),$$

$$\mathrm{i}(\mathbf{d}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{d}}\mathbf{c}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{c}}) = 2(d_1\mathbf{b}_s - b_1\mathbf{d}_s - \mathbf{b}_s\times\mathbf{d}_s) + 2(c_1\mathbf{a}_s - a_1\mathbf{c}_s + \mathbf{c}_s\times\mathbf{a}_s),$$

and the second equalities for $\mathbf{G}_1^s$, $\mathbf{G}_5^s$ and $\mathbf{G}_0^s$ as that appeared in Eqs. (B.31), (B.33) and (B.34) are proved.

It remains to check the validity of the second equalities for $\mathbf{G}_s^1$, $\mathbf{G}_s^5$, and $\mathbf{G}_s^0$. Similarly, through some calculations we have

$$\mathbf{d}\langle\bar{\mathbf{a}}\mathrm{i}\mathbf{x}\rangle = \mathbf{d}_s\times\mathbf{a}_s\cdot\mathbf{x}_s, \tag{B.51}$$

and

$$\mathbf{d}\langle\bar{\mathbf{a}}\mathrm{i}\rangle = \mathbf{d}_s\times\mathbf{a}_s \tag{B.52}$$

can be viewed as a row vector, since it linearly maps $\mathbf{x}$ to a scalar. With this formula the following relations are obvious

$$\mathbf{b}\langle\bar{\mathbf{c}}\mathrm{i}\rangle = \mathbf{b}_s\times\mathbf{c}_s, \quad \mathbf{c}\langle\bar{\mathbf{b}}\mathrm{i}\rangle = \mathbf{c}_s\times\mathbf{b}_s, \quad \mathbf{a}\langle\bar{\mathbf{d}}\mathrm{i}\rangle = \mathbf{a}_s\times\mathbf{d}_s,$$

$$\mathbf{d}\langle\bar{\mathbf{c}}\mathrm{i}\rangle = \mathbf{d}_s\times\mathbf{c}_s, \quad \mathbf{b}\langle\bar{\mathbf{a}}\mathrm{i}\rangle = \mathbf{b}_s\times\mathbf{a}_s, \quad \mathbf{c}\langle\bar{\mathbf{d}}\mathrm{i}\rangle = \mathbf{c}_s\times\mathbf{d}_s, \quad \mathbf{a}\langle\bar{\mathbf{b}}\mathrm{i}\rangle = \mathbf{a}_s\times\mathbf{b}_s.$$

Substituting these formulae into Eqs. (B.28), (B.36) and (B.40), we obtain the second equalities for $\mathbf{G}_s^1$, $\mathbf{G}_s^5$, and $\mathbf{G}_s^0$. This completes the proof of Eqs. (B.27)–(B.42).

The above procedure for obtaining the spinor map from $SL(2,\mathbb{H})$ onto $SO_o(5,1)$ are illustrated in Fig. 1. This map is from the group $SL(2,\mathbb{H})$ onto the group $SO_o(5,1)$ and is a *spinor map* (two-valued representation) in the sense that the two spin transformations $\pm\mathbf{U}$ map to the same proper orthochronous Lorentz transformation $\mathbf{G}$. The above formulae require to know $\mathbf{U}(t)$. More definitely, we should know the time-varying Lie algebra $\mathbf{Q}(t)$, which render $\mathbf{U}(t)$ obtainable through the integration of differential equation $\dot{\mathbf{U}}(t) = \mathbf{Q}(t)\mathbf{U}(t)$. Thus, we below derive the conversion formulae from $\mathbf{Q} \in sl(2,\mathbb{H})$ to $\mathbf{A} \in so(5,1)$, and vice versa.

Appendix C. The explicit isomorphism of $sl(2,\mathbb{H})$ and $so(5,1)$

Now, we attempt to convert the six-dimensional system (56) to a corresponding quaternion system with dimensions two, that is

$$\dot{\mathbf{U}}(t) = \mathbf{Q}(t)\mathbf{U}(t), \tag{C.1}$$

$$\mathbf{U}(0) = \mathbf{I}_2, \tag{C.2}$$

in which

$$\mathbf{Q} = \begin{bmatrix} \mathbf{r} & \mathbf{q} \\ \mathbf{p} & \mathbf{o} \end{bmatrix} \tag{C.3}$$

is a quaternionic matrix, subject to $\mathrm{Sca}(\mathbf{r}+\mathbf{o}) = 0$. The conversion relation is indeed a Lie algebra isomorphism of $sl(2,\mathbb{H}) \ni \mathbf{Q}$ onto $so(5,1) \ni \mathbf{A}$.

Parametrizing the spin transformation $\mathbf{U} \in SL(2,\mathbb{H})$

$$\mathbf{H}(t) = \mathbf{U}(t)\mathbf{H}(0)\overline{\mathbf{U}}^{\mathrm{T}}(t), \tag{C.4}$$

differentiating Eq. (C.4) with respect to t, and using Eq. (C.1) we thus obtain

$$\begin{bmatrix} \dot{x}^+ & \dot{\bar{\mathbf{x}}} \\ \dot{\mathbf{x}} & \dot{x}^- \end{bmatrix} = \begin{bmatrix} \mathbf{r} & \mathbf{q} \\ \mathbf{p} & \mathbf{o} \end{bmatrix} \begin{bmatrix} x^+ & \bar{\mathbf{x}} \\ \mathbf{x} & x^- \end{bmatrix} + \begin{bmatrix} x^+ & \bar{\mathbf{x}} \\ \mathbf{x} & x^- \end{bmatrix} \begin{bmatrix} \bar{\mathbf{r}} & \bar{\mathbf{p}} \\ \bar{\mathbf{q}} & \bar{\mathbf{o}} \end{bmatrix}. \tag{C.5}$$

It can be written as

$$\begin{bmatrix} \dot{x}^+ \\ \dot{\bar{\mathbf{x}}} \\ \dot{\mathbf{x}} \\ \dot{x}^- \end{bmatrix} = \begin{bmatrix} x^+(\mathbf{r}+\bar{\mathbf{r}}) + \mathbf{q}\mathbf{x} + \bar{\mathbf{x}}\bar{\mathbf{q}} \\ x^-\mathbf{q} + x^+\bar{\mathbf{p}} + \mathbf{r}\bar{\mathbf{x}} + \bar{\mathbf{x}}\bar{\mathbf{o}} \\ x^+\mathbf{p} + x^-\bar{\mathbf{q}} + \mathbf{o}\mathbf{x} + \mathbf{x}\bar{\mathbf{r}} \\ x^-(\mathbf{o}+\bar{\mathbf{o}}) + \mathbf{p}\bar{\mathbf{x}} + \mathbf{x}\bar{\mathbf{p}} \end{bmatrix}. \tag{C.6}$$

From Eq. (B.12) it follows that

$$\bar{\mathbf{x}}\bar{\mathbf{q}} = \bar{\mathbf{q}}\bar{\mathbf{x}} - 2\langle\bar{\mathbf{q}}\bar{\mathbf{x}}\rangle,$$

$$\bar{\mathbf{x}}\bar{\mathbf{o}} = \bar{\mathbf{o}}\bar{\mathbf{x}} - 2\langle\bar{\mathbf{o}}\bar{\mathbf{x}}\rangle,$$

$$\mathbf{x}\bar{\mathbf{r}} = \bar{\mathbf{r}}\mathbf{x} - 2\langle\bar{\mathbf{r}}\mathbf{x}\rangle,$$

$$\mathbf{x}\bar{\mathbf{p}} = \bar{\mathbf{p}}\mathbf{x} - 2\langle\bar{\mathbf{p}}\mathbf{x}\rangle.$$

Substituting these equations into Eq. (C.6) yields

$$\begin{bmatrix} \dot{x}^+ \\ \dot{\bar{\mathbf{x}}} \\ \dot{\mathbf{x}} \\ \dot{x}^- \end{bmatrix} = \begin{bmatrix} x^+(\mathbf{r}+\bar{\mathbf{r}}) + \mathbf{q}\mathbf{x} + \bar{\mathbf{q}}\bar{\mathbf{x}} - 2\langle\bar{\mathbf{q}}\bar{\mathbf{x}}\rangle \\ x^-\mathbf{q} + x^+\bar{\mathbf{p}} + \mathbf{r}\bar{\mathbf{x}} + \bar{\mathbf{o}}\bar{\mathbf{x}} - 2\langle\bar{\mathbf{o}}\bar{\mathbf{x}}\rangle \\ x^+\mathbf{p} + x^-\bar{\mathbf{q}} + \mathbf{o}\mathbf{x} + \bar{\mathbf{r}}\mathbf{x} - 2\langle\bar{\mathbf{r}}\mathbf{x}\rangle \\ x^-(\mathbf{o}+\bar{\mathbf{o}}) + \mathbf{p}\bar{\mathbf{x}} + \bar{\mathbf{p}}\mathbf{x} - 2\langle\bar{\mathbf{p}}\mathbf{x}\rangle \end{bmatrix}, \tag{C.7}$$

which can be rearranged to

$$\begin{bmatrix} \dot{x}^+ \\ \dot{\bar{\mathbf{x}}} \\ \dot{\mathbf{x}} \\ \dot{x}^- \end{bmatrix} = \begin{bmatrix} \mathbf{r}+\bar{\mathbf{r}} & \bar{\mathbf{q}} - 2\langle\bar{\mathbf{q}} & \mathbf{q} & 0 \\ \bar{\mathbf{p}} & \mathbf{r}+\bar{\mathbf{o}} - 2\langle\bar{\mathbf{o}} & 0 & \mathbf{q} \\ \mathbf{p} & 0 & \mathbf{o}+\bar{\mathbf{r}} - 2\langle\bar{\mathbf{r}} & \bar{\mathbf{q}} \\ 0 & \mathbf{p} & \bar{\mathbf{p}} - 2\langle\bar{\mathbf{p}} & \mathbf{o}+\bar{\mathbf{o}} \end{bmatrix} \begin{bmatrix} x^+ \\ \bar{\mathbf{x}} \\ \mathbf{x} \\ x^- \end{bmatrix}. \tag{C.8}$$

Also parametrizing Eq. (B.17) as

$$\begin{bmatrix} x^+(t) \\ \bar{\mathbf{x}}(t) \\ \mathbf{x}(t) \\ x^-(t) \end{bmatrix} = \mathbf{J}(t) \begin{bmatrix} x^+(0) \\ \bar{\mathbf{x}}(0) \\ \mathbf{x}(0) \\ x^-(0) \end{bmatrix}, \tag{C.9}$$

taking the time derivative of Eq. (C.9) and then using Eq. (C.9) again, we have

$$\begin{bmatrix} \dot{x}^+ \\ \dot{\bar{\mathbf{x}}} \\ \dot{\mathbf{x}} \\ \dot{x}^- \end{bmatrix} = \dot{\mathbf{J}}\mathbf{J}^{-1} \begin{bmatrix} x^+ \\ \bar{\mathbf{x}} \\ \mathbf{x} \\ x^- \end{bmatrix}. \tag{C.10}$$

Comparing the above equation with Eq. (C.8) yields

$$\dot{\mathbf{J}}\mathbf{J}^{-1} = \begin{bmatrix} \mathbf{r}+\bar{\mathbf{r}} & \bar{\mathbf{q}} - 2\langle\bar{\mathbf{q}} & \mathbf{q} & 0 \\ \bar{\mathbf{p}} & \mathbf{r}+\bar{\mathbf{o}} - 2\langle\bar{\mathbf{o}} & 0 & \mathbf{q} \\ \mathbf{p} & 0 & \mathbf{o}+\bar{\mathbf{r}} - 2\langle\bar{\mathbf{r}} & \bar{\mathbf{q}} \\ 0 & \mathbf{p} & \bar{\mathbf{p}} - 2\langle\mathbf{p} & \mathbf{o}+\bar{\mathbf{o}} \end{bmatrix}. \tag{C.11}$$

For the proper orthochronous Lorentz transformation $\mathbf{G} \in SO_o(5,1)$, taking the time derivative of Eq. (B.25), and using Eqs. (56), (C.11) and (B.25) again, we obtain

$$\mathbf{A} = \mathbf{C}^{-1} \begin{bmatrix} \mathbf{r}+\bar{\mathbf{r}} & \bar{\mathbf{q}} - 2\langle\bar{\mathbf{q}} & \mathbf{q} & 0 \\ \bar{\mathbf{p}} & \mathbf{r}+\bar{\mathbf{o}} - 2\langle\bar{\mathbf{o}} & 0 & \mathbf{q} \\ \mathbf{p} & 0 & \mathbf{o}+\bar{\mathbf{r}} - 2\langle\bar{\mathbf{r}} & \bar{\mathbf{q}} \\ 0 & \mathbf{p} & \bar{\mathbf{p}} - 2\langle\bar{\mathbf{p}} & \mathbf{o}+\bar{\mathbf{o}} \end{bmatrix} \mathbf{C}. \tag{C.12}$$

Letting

$$\mathbf{A} := \begin{bmatrix} A_{11} & \mathbf{A}_{1s} & A_{15} & A_{10} \\ \mathbf{A}_{s1} & \mathbf{A}_{ss} & \mathbf{A}_{s5} & \mathbf{A}_{s0} \\ A_{51} & \mathbf{A}_{5s} & A_{55} & A_{50} \\ A_{01} & \mathbf{A}_{0s} & A_{05} & A_{00} \end{bmatrix}, \tag{C.13}$$

and then substituting Eq. (B.23) for $\mathbf{C}^{-1}$ and Eq. (B.21) for $\mathbf{C}$ into Eq. (C.12), we obtain each term in $\mathbf{A}$ as follows:

$$A_{11} = \frac{1}{2}(\mathbf{r}+\bar{\mathbf{r}}+\mathbf{o}+\bar{\mathbf{o}}) - \langle\bar{\mathbf{o}}1\rangle - \langle\bar{\mathbf{r}}1\rangle, \tag{C.14}$$

$$\mathbf{A}_{1s} = \frac{\mathrm{i}}{2}(\bar{\mathbf{r}} - \mathbf{r} + \mathbf{o} - \bar{\mathbf{o}}) + \langle\bar{\mathbf{o}}\mathrm{i}\rangle - \langle\bar{\mathbf{r}}\mathrm{i}\rangle, \tag{C.15}$$

$$A_{15} = \frac{1}{2}(\mathbf{p} + \bar{\mathbf{p}} - \mathbf{q} - \bar{\mathbf{q}}), \tag{C.16}$$

$$A_{10} = \frac{1}{2}(\mathbf{p} + \bar{\mathbf{p}} + \mathbf{q} + \bar{\mathbf{q}}), \tag{C.17}$$

$$\mathbf{A}_{s1} = \frac{\mathrm{i}}{2}(\mathbf{r} - \bar{\mathbf{r}} + \bar{\mathbf{o}} - \mathbf{o}) - \mathrm{i}\langle\bar{\mathbf{o}}1\rangle + \mathrm{i}\langle\bar{\mathbf{r}}1\rangle, \tag{C.18}$$

$$\mathbf{A}_{ss} = \frac{1}{2}(\mathbf{r} + \bar{\mathbf{r}} + \mathbf{o} + \bar{\mathbf{o}}) + \mathrm{i}\langle\bar{\mathbf{o}}\mathrm{i}\rangle + \mathrm{i}\langle\bar{\mathbf{r}}\mathrm{i}\rangle, \tag{C.19}$$

$$\mathbf{A}_{s5} = \frac{\mathrm{i}}{2}(\bar{\mathbf{p}} - \mathbf{p} - \mathbf{q} + \bar{\mathbf{q}}), \tag{C.20}$$

$$\mathbf{A}_{s0} = \frac{\mathrm{i}}{2}(\bar{\mathbf{p}} - \mathbf{p} + \mathbf{q} - \bar{\mathbf{q}}), \tag{C.21}$$

$$A_{51} = \frac{1}{2}(\mathbf{q} + \bar{\mathbf{q}} - \mathbf{p} - \bar{\mathbf{p}}) - \langle\bar{\mathbf{q}}1\rangle + \langle\bar{\mathbf{p}}1\rangle, \tag{C.22}$$

$$\mathbf{A}_{5s} = \frac{\mathrm{i}}{2}(\mathbf{p} - \bar{\mathbf{p}} + \mathbf{q} - \bar{\mathbf{q}}) + \langle\bar{\mathbf{q}}\mathrm{i}\rangle + \langle\bar{\mathbf{p}}\mathrm{i}\rangle, \tag{C.23}$$

$$A_{55} = \frac{1}{2}(\mathbf{r} + \bar{\mathbf{r}} + \mathbf{o} + \bar{\mathbf{o}}), \tag{C.24}$$

$$A_{50} = \frac{1}{2}(\mathbf{r} + \bar{\mathbf{r}} - \mathbf{o} - \bar{\mathbf{o}}), \tag{C.25}$$

$$A_{01} = \frac{1}{2}(\mathbf{p} + \bar{\mathbf{p}} + \mathbf{q} + \bar{\mathbf{q}}) - \langle\bar{\mathbf{q}}1\rangle - \langle\bar{\mathbf{p}}1\rangle, \tag{C.26}$$

$$\mathbf{A}_{0s} = \frac{\mathrm{i}}{2}(\mathbf{q} - \bar{\mathbf{q}} + \bar{\mathbf{p}} - \mathbf{p}) + \langle\bar{\mathbf{q}}\mathrm{i}\rangle - \langle\bar{\mathbf{p}}\mathrm{i}\rangle, \tag{C.27}$$

$$A_{05} = \frac{1}{2}(\mathbf{r} + \bar{\mathbf{r}} - \mathbf{o} - \bar{\mathbf{o}}), \tag{C.28}$$

$$A_{00} = \frac{1}{2}(\mathbf{r} + \bar{\mathbf{r}} + \mathbf{o} + \bar{\mathbf{o}}). \tag{C.29}$$

First, we note due to $\mathrm{Sca}(\mathbf{r} + \mathbf{o}) = 0$ that

$$\mathbf{r} + \bar{\mathbf{r}} + \mathbf{o} + \bar{\mathbf{o}} = 2\mathrm{Sca}(\mathbf{r} + \mathbf{o}) = 0. \tag{C.30}$$

It thus follows that $A_{55} = A_{00} = 0$. Second, we show through some calculations that

$$\mathrm{i}\langle\bar{\mathbf{o}}\mathrm{i}\mathbf{x}\rangle + \mathrm{i}\langle\bar{\mathbf{r}}\mathrm{i}\mathbf{x}\rangle = (\tilde{\mathbf{o}}_{\mathrm{s}} + \tilde{\mathbf{r}}_{\mathrm{s}})\mathbf{x}^{\mathrm{s}} \tag{C.31}$$

and thus,

$$\mathrm{i}\langle\bar{\mathbf{o}}\mathrm{i}\rangle + \mathrm{i}\langle\bar{\mathbf{r}}\mathrm{i}\rangle = \tilde{\mathbf{o}}_\mathrm{s} + \tilde{\mathbf{r}}_\mathrm{s}. \tag{C.32}$$

Substituting Eqs. (C.30) and (C.32) into Eq. (C.19) gives $\mathbf{A}_{ss} = \tilde{\mathbf{o}}_\mathrm{s} + \tilde{\mathbf{r}}_\mathrm{s}$. Now, by using

$$\langle\bar{\mathbf{o}}1\rangle = 0, \quad \langle\bar{\mathbf{p}}1\rangle = 0, \quad \langle\bar{\mathbf{q}}1\rangle = 0, \quad \langle\bar{\mathbf{r}}1\rangle = 0,$$

$$\langle\bar{\mathbf{o}}\mathrm{i}\rangle = 0, \quad \langle\bar{\mathbf{p}}\mathrm{i}\rangle = 0, \quad \langle\bar{\mathbf{q}}\mathrm{i}\rangle = 0, \quad \langle\bar{\mathbf{r}}\mathrm{i}\rangle = 0$$

on the other equations in Eqs. (C.14)–(C.29), and noting that

$$\mathbf{x} + \bar{\mathbf{x}} = 2x^1 = 2\mathrm{Sca}(\mathbf{x}), \quad \mathbf{x} - \bar{\mathbf{x}} = 2\mathrm{i}\mathbf{x}^\mathrm{s} = 2\mathrm{i}\mathrm{Vec}(\mathbf{x}),$$

we eventually obtain

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{r}_\mathrm{s}^\mathrm{T} - \mathbf{o}_\mathrm{s}^\mathrm{T} & p_1 - q_1 & p_1 + q_1, \\ \mathbf{o}_\mathrm{s} - \mathbf{r}_\mathrm{s} & \tilde{\mathbf{o}}_\mathrm{s} + \tilde{\mathbf{r}}_\mathrm{s} & \mathbf{p}_\mathrm{s} + \mathbf{q}_\mathrm{s} & \mathbf{p}_\mathrm{s} - \mathbf{q}_\mathrm{s} \\ q_1 - p_1 & -\mathbf{p}_\mathrm{s}^\mathrm{T} - \mathbf{q}_\mathrm{s}^\mathrm{T} & 0 & r_1 - o_1 \\ p_1 + q_1 & \mathbf{p}_\mathrm{s}^\mathrm{T} - \mathbf{q}_\mathrm{s}^\mathrm{T} & r_1 - o_1 & 0 \end{bmatrix}. \tag{C.33}$$

The above formula enables us to obtain **A** from **Q**. It is easy to check that such **A**, satisfying Eq. (55), is an element of the Lie algebra $so(5,1)$.

In order to derive **Q** from **A**, we let

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{A}_{1s}^\mathrm{T} & A_{15} & A_{10} \\ -\mathbf{A}_{1s} & \mathbf{A}_{ss} & \mathbf{A}_{s5} & \mathbf{A}_{s0} \\ -A_{15} & -\mathbf{A}_{s5}^\mathrm{T} & 0 & A_{50} \\ A_{10} & \mathbf{A}_{s0}^\mathrm{T} & A_{50} & 0 \end{bmatrix} \tag{C.34}$$

by considering the 6×6 real matrix $\mathbf{A} \in so(5,1)$. In above $\mathbf{A}_{ss}$ is a 3×3 skew–symmetric matrix function. Multiplying Eq. (C.12) by **C** from the left-hand side, then substituting Eq. (C.34) for **A** and Eq. (B.21) for **C** into the resultant, we obtain

$$\begin{aligned} &\begin{bmatrix} \mathbf{r}+\bar{\mathbf{r}} & \bar{\mathbf{q}} - 2\langle\bar{\mathbf{q}} & \mathbf{q} & 0 \\ \bar{\mathbf{p}} & \mathbf{r}+\bar{\mathbf{o}} - 2\langle\bar{\mathbf{o}} & 0 & \mathbf{q} \\ \mathbf{p} & 0 & \mathbf{o}+\bar{\mathbf{r}} - 2\langle\bar{\mathbf{r}} & \bar{\mathbf{q}} \\ 0 & \mathbf{p} & \bar{\mathbf{p}} - 2\langle\bar{\mathbf{p}} & \mathbf{o}+\bar{\mathbf{o}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & -\mathrm{i} & 0 & 0 \\ 1 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & -\mathrm{i} & 0 & 0 \\ 1 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{A}_{1s}^\mathrm{T} & A_{15} & A_{10} \\ -\mathbf{A}_{1s} & \mathbf{A}_{ss} & \mathbf{A}_{s5} & \mathbf{A}_{s0} \\ -A_{15} & -\mathbf{A}_{s5}^\mathrm{T} & 0 & A_{50} \\ A_{10} & \mathbf{A}_{s0}^\mathrm{T} & A_{50} & 0 \end{bmatrix}. \end{aligned} \tag{C.35}$$

Expanding the above quaternion algebraic equation generates

$$\begin{bmatrix} \mathbf{q}+\bar{\mathbf{q}} & \mathrm{i}(\mathbf{q}-\bar{\mathbf{q}}) & \mathbf{r}+\bar{\mathbf{r}} & \mathbf{r}+\bar{\mathbf{r}} \\ \mathbf{r}+\bar{\mathbf{o}} & -\mathrm{i}(\mathbf{r}+\bar{\mathbf{o}}+2\tilde{\mathbf{o}}_\mathrm{s}) & \bar{\mathbf{p}}-\mathbf{q} & \bar{\mathbf{p}}+\mathbf{q} \\ \mathbf{o}+\bar{\mathbf{r}} & \mathrm{i}(\mathbf{o}+\bar{\mathbf{r}}+2\tilde{\mathbf{r}}_\mathrm{s}) & \mathbf{p}-\bar{\mathbf{q}} & \mathbf{p}+\bar{\mathbf{q}} \\ \mathbf{p}+\bar{\mathbf{p}} & \mathrm{i}(\bar{\mathbf{p}}-\mathbf{p}) & -\mathbf{o}-\bar{\mathbf{o}} & \mathbf{o}+\bar{\mathbf{o}} \end{bmatrix} = \begin{bmatrix} A_{10}-A_{15} & \mathbf{A}_{s0}^\mathrm{T} - \mathbf{A}_{s5}^\mathrm{T} & A_{50} & A_{50} \\ \mathrm{i}\mathbf{A}_{1s} & \mathbf{A}_{1s}^\mathrm{T} - \mathrm{i}\mathbf{A}_{ss} & A_{15} - \mathrm{i}\mathbf{A}_{s5} & A_{10} - \mathrm{i}\mathbf{A}_{s0} \\ -\mathrm{i}\mathbf{A}_{1s} & \mathbf{A}_{1s}^\mathrm{T} + \mathrm{i}\mathbf{A}_{ss} & A_{15} + \mathrm{i}\mathbf{A}_{s5} & A_{10} + \mathrm{i}\mathbf{A}_{s0} \\ A_{10}+A_{15} & \mathbf{A}_{s0}^\mathrm{T} + \mathbf{A}_{s5}^\mathrm{T} & A_{50} & -A_{50} \end{bmatrix}, \tag{C.36}$$

where $\langle\bar{\mathbf{o}}\mathrm{i}\rangle = -\mathrm{i}\tilde{\mathbf{o}}_\mathrm{s}$, $\langle\bar{\mathbf{r}}\mathrm{i}\rangle = -\mathrm{i}\tilde{\mathbf{r}}_\mathrm{s}$, $\langle\bar{\mathbf{o}}1\rangle = 0$, $\langle\bar{\mathbf{q}}1\rangle = 0$, $\langle\bar{\mathbf{r}}1\rangle = 0$, $\langle\bar{\mathbf{p}}1\rangle = 0$, $\langle\bar{\mathbf{p}}\mathrm{i}\rangle = 0$, and $\langle\bar{\mathbf{q}}\mathrm{i}\rangle = 0$ were used.

From Eq. (C.36) we obtain

$$\mathbf{Q} = \begin{bmatrix} \mathbf{r} & \mathbf{q} \\ \mathbf{p} & \mathbf{o} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A_{50} + \mathrm{i}(\mathrm{axial}(\mathbf{A}_{ss}) + \mathbf{A}_{1s}) & A_{10} - A_{15} + \mathrm{i}(\mathbf{A}_{s5} - \mathbf{A}_{s0}) \\ A_{10} + A_{15} + \mathrm{i}(\mathbf{A}_{s5} + \mathbf{A}_{s0}) & -A_{50} + \mathrm{i}(\mathrm{axial}(\mathbf{A}_{ss}) - \mathbf{A}_{1s}) \end{bmatrix}, \tag{C.37}$$

in which axial $(\mathbf{A}_{ss})$ denotes the axial vector of $\mathbf{A}_{ss}$, that is, $(\mathrm{axial}(\mathbf{A}_{ss}))_i = \frac{1}{2}\epsilon_{ijk}(\mathbf{A}_{ss})_{kj}$, where ϵ_{ijk} is the Levi–Civita permutation symbol. It is obvious that $\mathbf{Q}$ thus obtained is an element of $sl(2,\mathbb{H})$. Formula (C.37) enables us to obtain $\mathbf{Q}$ from $\mathbf{A}$. Formulae (C.13) and (C.37) explicitly expressing the Lie algebras isomorphism of $sl(2,\mathbb{H})$ and $so(5,1)$ are very important for further calculations. Both the Lie algebras $sl(2,\mathbb{H})$ and $so(5,1)$ have 15 independent parameters.

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